

A NOTE ON THE VARIANCE OF THE SQUARE COMPONENTS OF A NORMAL MULTIVARIATE WITHIN A EUCLIDEAN BALL

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Abstract. We present arguments in favour of the inequalities $\text{var}(X_n^2 | X \in \mathcal{B}_v(\rho)) \leq 2\lambda_n \mathbb{E}[X_n^2 | X \in \mathcal{B}_v(\rho)]$, where $X \sim \mathcal{N}_v(0, \Lambda)$ is a normal vector in $v \geq 1$ dimensions, with zero mean and covariance matrix $\Lambda = \text{diag}(\lambda)$, and $\mathcal{B}_v(\rho)$ is a centered v -dimensional Euclidean ball of square radius ρ . Such relations lie at the heart of an iterative algorithm, proposed in ref. [1] to perform a reconstruction of Λ from the covariance matrix of X conditioned to $\mathcal{B}_v(\rho)$. In the regime of strong truncation, *i.e.* for $\rho \lesssim \lambda_n$, the above inequality is easily proved, whereas it becomes harder for $\rho \gg \lambda_n$. Here, we expand both sides in a function series controlled by powers of λ_n/ρ and show that the coefficient functions of the series fulfill the inequality order by order if ρ is sufficiently large. The intermediate region remains at present an open challenge.

1 Introduction

It is intuitively clear that independent random variables develop correlations once constrained within compact multivariate domains. Whenever the mathematical framework rules out closed-form results, a possible approach to studying such correlations is to focus on inequalities among expected values. As a case in point, in this paper we consider a random vector $X \sim \mathcal{N}_v(0, \Lambda)$ in $v \geq 1$ dimensions, with $\Lambda = \text{diag}(\lambda)$ and $\lambda = \{\lambda_k\}_{k=1}^v$, whose probability density is cut off sharply outside a Euclidean ball

$$\mathcal{B}_v(\rho) = \{x \in \mathbb{R}^v : x^T x < \rho\}. \quad (1.1)$$

Owing to the symmetry mismatch between $\mathcal{N}_v(0, \Lambda)$ and $\mathcal{B}_v(\rho)$, the conditional moments of X admit no exact representation in terms of elementary functions. Our aim is to show that the effect of the spherical truncation on the variance of the square components of X is quantified by the inequalities

$$\Delta_n(\rho; \lambda) \equiv \frac{1}{\rho^2} \left\{ \text{var}(X_n^2 | X \in \mathcal{B}_v(\rho)) - 2\lambda_n \mathbb{E}[X_n^2 | X \in \mathcal{B}_v(\rho)] \right\} \leq 0, \quad n = 1, \dots, v. \quad (1.2)$$

The interest we have in eq. (1.2) originates from ref. [1], where we have proposed a fixed-point algorithm for the reconstruction of Λ , in case the only available information amounts to the covariance matrix $\mathfrak{S}_{\mathcal{B}}$ of X conditioned to $\mathcal{B}_v(\rho)$. In particular, in that paper we showed that eq. (1.2) and the correlation inequalities

$$\text{cov}(X_n^2, X_m^2 | X \in \mathcal{B}_v(\rho)) \leq 0, \quad n \neq m, \quad (1.3)$$

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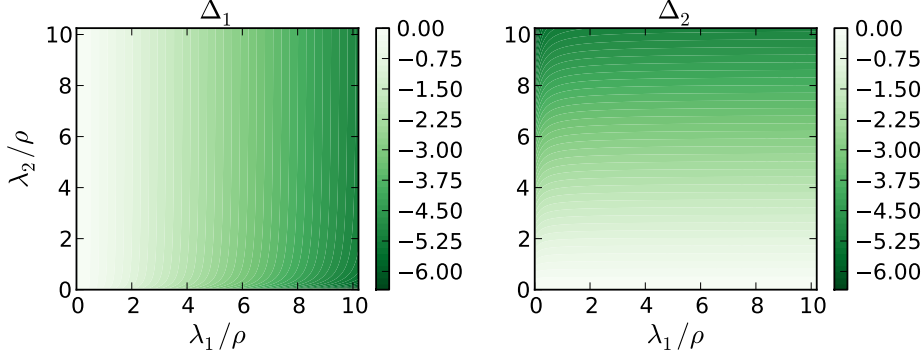


Fig. 1 – Contour plots of Δ_n at $v = 2$.

are necessary and sufficient for the convergence of the algorithm. Eq. (1.3) expresses a property of negative association among the square components of X (see ref. [2]). A proof of it goes beyond the scope of the present paper.

If we denote Gaussian integrals over Euclidean balls by

$$\alpha_{k\ell m\dots}(\rho; \lambda) = \int_{\mathcal{B}_v(\rho)} d^v x \frac{x_k^2}{\lambda_k} \frac{x_\ell^2}{\lambda_\ell} \frac{x_m^2}{\lambda_m} \dots \prod_{j=1}^v \delta(x_j, \lambda_j), \quad \delta(y, \eta) = \frac{e^{-y^2/(2\eta)}}{(2\pi\eta)^{1/2}}, \quad (1.4)$$

and define $\partial_n \equiv \partial/(\partial\lambda_n)$, we see that $\Delta_n = 2(\lambda_n^2/\rho^2)[\lambda_n \partial_n(\alpha_n/\alpha)]$. Thus, the inequality $\Delta_n \leq 0$ holds true iff α_n/α is monotonic decreasing in λ_n with $\lambda_{(n)} \equiv \{\lambda_i\}_{i \neq n}$ kept fixed. Since such monotonic behavior is held by both α_n and α separately, eq. (1.2) simply means that α_n is more rapidly decreasing than α . An illustrative example is shown in Fig. 1, where contour plots of Δ_n at $v = 2$ are reported¹.

Apart from the covariance reconstruction algorithm, a different motivation to care about eqs. (1.2) and (1.3) has to do with non-linear optimization issues. Thanks to Prékopa's Theorem [3], $\alpha(\rho; \lambda)$ is easily shown to be logarithmic concave in ρ . In sect. 2 we discuss how log-concavity relates to the correlation inequalities. The outcome is that, independently of Prékopa's Theorem, eqs. (1.2) and (1.3) are alone sufficient to induce log-concavity, while they cannot be deduced from it.

Now, despite the seemingly candid aspect of eq. (1.2), it is difficult to find a unique rigorous proof of it, which works across the whole parameter space $(\rho; \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+^v$. In this paper we propose three different partial arguments. The first two are discussed in sect. 3. They are both straightforward and apply in the *regime of strong truncation*, i.e. for $0 < \rho < \lambda_n$ resp. $0 < \rho < 2\lambda_n$, independently of $\lambda_{(n)}$. In particular, the first one is based on Hölder's inequality, while the second one makes use of the integral representation of Δ_n . The relative ease of proving eq. (1.2) at strong truncation is certainly due to the large negative values Δ_n assumes in this regime and its weak dependence upon $\lambda_{(n)}$, which in a sense makes the problem *nearly* 1-dimensional.

The third argument, presented in sects. 4 and 5, applies instead in the *regime of weak truncation*, i.e. for $\rho \gg \lambda_n$, where proving eq. (1.2) is definitely harder. As $\rho \rightarrow \infty$, we have indeed $\text{var}(X_n^2 | X \in \mathcal{B}_v(\rho)) \rightarrow 2\lambda_n^2$ and $\mathbb{E}[X_n^2 | X \in \mathcal{B}_v(\rho)] \rightarrow \lambda_n$, thus $\Delta_n \rightarrow 0$. Hence, if eq. (1.2) is correct, it must follow from a cancellation of two positive terms resulting in an increasingly small negative balance. Motivated by the observation that the volume constraint weakens as $\rho \rightarrow \infty$ (and consequently $\alpha_{k\ell m\dots}$ becomes well approximated by a product of 1-dimensional Gaussian integrals), we expand Δ_n in a non-elementary-function series around the factorization point. Each term of the expansion factorizes into a 1-dimensional integral along the n^{th} direction plus a residual $(v-1)$ -dimensional integral in the orthogonal subspace. We prove that such factors get opposite signs as $\rho \rightarrow \infty$, thus resulting in negative contributions.

¹Numerical techniques for the computation of $\alpha_{k\ell m\dots}$ are discussed in ref.[1].

We finally draw our conclusions in sect. 6.

2 Basic properties of $\alpha_{k\ell m\dots}$

It is worthwhile starting our discussion by reviewing some trivial properties of the Gaussian integrals, which are used in the sequel. As an alternative notation for $\alpha_{k\ell m\dots}$ we sometimes adopt the symbol $\alpha_{1:k_1\dots v:k_v}$, where k_j counts the multiplicity of the index $j = 1, \dots, v$. Whenever a directional index has zero multiplicity, we simply drop it. For instance, we write $\alpha_{j:k_j}$ in place of the more pedantic $\alpha_{1:0\dots j:k_j\dots v:0}$. When needed, we declare the integral dimension of $\alpha_{k\ell m\dots}$ explicitly by writing the latter as $\alpha_{k\ell m\dots}^{(v)}$. With this in mind, we proceed to a first set of statements:

Proposition 2.1. *Gaussian integrals fulfill the following properties:*

$p_1)$ $\alpha_{k\ell m\dots}(\rho; \lambda)$ is increasing in ρ ;

$p_2)$ $\alpha_{k\ell m\dots}(\rho; \lambda)$ is separately decreasing in $\lambda_1, \dots, \lambda_v$;

$p_3)$ $\alpha_{k\ell m\dots}(\rho; \lambda)$ fulfills the scaling equation

$$\left[\rho \partial_\rho + \sum_{r=1}^v \lambda_r \partial_r \right] \alpha_{k\ell m\dots}(\rho; \lambda) = 0; \quad (2.1)$$

$p_4)$ one-index integrals $\alpha_{k:n}$ follow the hierarchy

$$\alpha_{k:n} \leq (2n-1)\alpha_{k:(n-1)} \leq (2n-1)(2n-3)\alpha_{k:(n-2)} \leq \dots \leq (2n-1)!! \alpha \leq (2n-1)!!; \quad (2.2)$$

$p_5)$ differentiating $\alpha_{k\ell m\dots}(\rho; \lambda)$ with respect to ρ yields

$$\rho \partial_\rho \alpha_{k_1\dots k_n}(\rho; \lambda) = \frac{1}{2}(v+2n)\alpha_{k_1\dots k_n}(\rho; \lambda) - \frac{1}{2} \sum_{k=1}^v \alpha_{k_1\dots k_n k}(\rho; \lambda), \quad n = 0, 1, 2, \dots \quad (2.3)$$

$p_6)$ $\alpha(\rho; \lambda)$ is logarithmic concave in ρ , i.e. it fulfills

$$\alpha(s\rho_1 + (1-s)\rho_2; \lambda) \geq [\alpha(\rho_1; \lambda)]^s [\alpha(\rho_2; \lambda)]^{1-s}, \quad 0 \leq s \leq 1, \quad \rho_1, \rho_2 \in \mathbb{R}_+. \quad (2.4)$$

Proof. Property $p_1)$ follows from the positiveness of the integrand of $\alpha_{k\ell m\dots}$ and the observation that if $\rho_1 < \rho_2$ then $\mathcal{B}_v(\rho_1) \subset \mathcal{B}_v(\rho_2)$. \square Property $p_2)$ follows from the observation that $\alpha_{k\ell m\dots}$ depends on ρ and λ only via adimensional ratios, as seen by rescaling the integration variable $x \rightarrow x/\sqrt{\rho}$ in eq. (1.4), i.e.

$$\begin{aligned} \alpha_{k\ell m\dots}(\rho; \lambda) &= \frac{\rho^{v/2}}{(2\pi)^{v/2} |\Lambda|^{1/2}} \int_{\mathcal{B}_v(1)} d^v x \frac{\rho x_k^2}{\lambda_k} \frac{\rho x_\ell^2}{\lambda_\ell} \frac{\rho x_m^2}{\lambda_m} \dots \exp \left\{ -\frac{\rho}{2} \sum_{m=1}^v \frac{x_m^2}{\lambda_m} \right\} \\ &= \alpha_{k\ell m\dots} \left(1; \left\{ \frac{\lambda_1}{\rho}, \dots, \frac{\lambda_v}{\rho} \right\} \right). \end{aligned} \quad (2.5)$$

When a single variance is downscaled, e.g. $\lambda \rightarrow \lambda' = \{\lambda_1, \dots, a\lambda_r, \dots, \lambda_v\}$ with $0 < a < 1$, the change in $\alpha_{k\ell m\dots}$ is entirely transferred to the integration region, i.e.

$$\alpha_{k\ell m\dots}(\rho; \lambda') = \frac{\rho^{v/2}}{(2\pi)^{v/2} |\Lambda|^{1/2}} \int_{\mathcal{E}_v(1;a)} d^v x \frac{\rho x_k^2}{\lambda_k} \frac{\rho x_\ell^2}{\lambda_\ell} \frac{\rho x_m^2}{\lambda_m} \dots \exp \left\{ -\frac{\rho}{2} \sum_{m=1}^v \frac{x_m^2}{\lambda_m} \right\}, \quad (2.6)$$

with

$$\mathcal{E}_v(1; a) = \{x \in \mathbb{R}^v : x_1^2 + \dots + ax_r^2 + \dots + x_v^2 < 1\} . \quad (2.7)$$

Since $\mathcal{B}_v(1) \subset \mathcal{E}_v(1; a)$, it follows $\alpha_{k\ell m\dots}(\rho; \lambda') > \alpha_{k\ell m\dots}(\rho; \lambda)$. \square Property p_3) follows from the application of the chain rule of differentiation to eq. (2.5). The meaning of the scaling equation is that $\alpha_{k\ell m\dots}$ keeps invariant under a change of the units adopted to measure both ρ and λ . \square To get convinced about property p_4), we first notice that

$$\lambda_k \partial_k \alpha_{1:n_1\dots k:n_k\dots v:n_v} = \frac{1}{2} [\alpha_{1:n_1\dots k:(n_k+1)\dots v:n_v} - (2n_k + 1) \alpha_{1:n_1\dots k:n_k\dots v:n_v}] , \quad (2.8)$$

as proved by evaluating the derivative on the *l.h.s.* under the integral sign. From property p_2), the *r.h.s.* of eq. (2.8) is recognized to be negative. The proof is completed by taking $n_j = 0$ for $j \neq k$. Note that eq. (2.2) entails the inequalities

$$\mathbb{E}[X_k^{2n} | X \in \mathcal{B}_v(\rho)] \leq (2n - 1)!! \lambda_k^n , \quad n = 1, 2, \dots , \quad (2.9)$$

with the quantity on the *r.h.s.* representing the value of the unconditioned $(2n)^{\text{th}}$ univariate moment of X_k . We shall use the lowest order inequalities $\mathbb{E}[X_k^2 | X \in \mathcal{B}_v(\rho)] \leq \lambda_k$ and $\mathbb{E}[X_k^4 | X \in \mathcal{B}_v(\rho)] \leq 3\lambda_k^2$ time and again in the sequel. Note also that the larger n , the slower $\alpha_{k:n}$ saturates to its infinite volume limit. Indeed, if we denote by $d_n \equiv [(2n - 1)!! - \alpha_{k:n}]/\alpha_{k:n}$ the fractional distance of $\alpha_{k:n}$ from its infinite volume limit, then eq. (2.2) is equivalent to the inequality chain

$$d_0(\rho; \lambda) \leq d_1(\rho; \lambda) \leq d_2(\rho; \lambda) \leq \dots . \quad (2.10)$$

As we shall see, this property lies at the heart of most of the difficulties related to proving eq. (1.2). \square Property p_5) follows from eqs. (2.1) and (2.8). \square Finally, in order to prove property p_6), we recall [3]

Theorem 2.1 (Prékopa). *Let $Q(x)$ be a convex function defined on the entire v -dimensional space \mathbb{R}^v . Suppose that $Q(x) \geq a$, where a is some real number. Let $\psi(z)$ be a function defined on the infinite interval $[a, \infty)$. Suppose that $\psi(z)$ is non-negative, non-increasing, differentiable, and $-\psi'(z)$ is logarithmic concave. Consider the function $f(x) = \psi(Q(x))$ ($x \in \mathbb{R}^v$) and suppose that it is a probability density, i.e.*

$$\int_{\mathbb{R}^v} d^v x f(x) = 1 \quad (2.11)$$

Denote by $P\{C\}$ the integral of $f(x)$ over the measurable subset C of \mathbb{R}^v . If A and B are any two convex sets in \mathbb{R}^v , then the following inequality holds:

$$(P\{A\})^s (P\{B\})^{1-s} \leq P\{sA + (1-s)B\} , \quad 0 \leq s \leq 1 , \quad (2.12)$$

where the linear combination on the l.h.s. denotes the Minkowski sum

$$sA + (1-s)B \equiv \{sx + (1-s)y : x \in A, y \in B\} . \quad (2.13)$$

Obviously, theorem 2.1 applies if $f(x)$ is a product of univariate Gaussian densities, as is the case with $\alpha(\rho; \lambda)$. In addition, if $x \in \mathcal{B}_v(\rho_1)$ and $y \in \mathcal{B}_v(\rho_2)$, from the convexity of the square function $x \mapsto x^2$ it follows that

$$\sum_{k=1}^v [sx_k + (1-s)y_k]^2 \leq s \sum_{k=1}^v x_k^2 + (1-s) \sum_{k=1}^v y_k^2 \leq s\rho_1 + (1-s)\rho_2 , \quad (2.14)$$

i.e. $s\mathcal{B}_v(\rho_1) + (1-s)\mathcal{B}_v(\rho_2) \subseteq \mathcal{B}_v(s\rho_1 + (1-s)\rho_2)$. Accordingly, we conclude that

$$\begin{aligned} [\alpha(\rho_1; \lambda)]^s [\alpha(\rho_2; \lambda)]^{1-s} &\leq \int_{s\mathcal{B}_v(\rho_1) + (1-s)\mathcal{B}_v(\rho_2)} d^v x \prod_{m=1}^v \delta(x_m, \lambda_m) \\ &\leq \int_{\mathcal{B}_v(s\rho_1 + (1-s)\rho_2)} d^v x \prod_{m=1}^v \delta(x_m, \lambda_m) = \alpha(s\rho_1 + (1-s)\rho_2; \lambda). \end{aligned} \quad (2.15)$$

□

Now, log-concavity is a local property of $\alpha(\rho; \lambda)$, yet it brings global information about the conditional moments of X . To see this, we observe that since $\alpha(\rho; \lambda)$ is twice differentiable with respect to ρ , eq. (2.4) is equivalent to

$$\alpha \partial_\rho^2 \alpha - (\partial_\rho \alpha)^2 \leq 0. \quad (2.16)$$

We iterate eq. (2.3) to express the above derivatives in terms of conditional expectations. In first place, evaluating that equation at $n = 0$ yields

$$\partial_\rho \alpha = \frac{v}{2\rho} \alpha - \frac{\alpha}{2\rho} \sum_{k=1}^v \frac{\mathbb{E}[X_k^2 | X \in \mathcal{B}_v(\rho)]}{\lambda_k}. \quad (2.17)$$

Property p_1) then implies

$$\sum_{k=1}^v \frac{\mathbb{E}[X_k^2 | X \in \mathcal{B}_v(\rho)]}{\lambda_k} \leq v. \quad (2.18)$$

Though trivial, eq. (2.18) calls for two remarks. The first one is that a sufficient (but not necessary) condition for it to hold true is $\mathbb{E}[X_k^2 | X \in \mathcal{B}_v(\rho)] \leq \lambda_k \quad \forall k$, which has already been established. In second place, differentiating α in ρ an arbitrary number of times generates always symmetric expressions with respect to the directional indices, since ρ is not tied to any specific direction. In particular, this is the case with the second derivative,

$$\partial_\rho^2 \alpha = \frac{\alpha}{\rho^2} \left\{ \frac{v(v-2)}{4} - \frac{v}{2} \sum_{k=1}^v \frac{\mathbb{E}[X_k^2 | X \in \mathcal{B}_v(\rho)]}{\lambda_k} + \frac{1}{4} \sum_{j,k=1}^v \frac{\mathbb{E}[X_j^2 X_k^2 | X \in \mathcal{B}_v(\rho)]}{\lambda_j \lambda_k} \right\}. \quad (2.19)$$

We see that all directional indices are again summed over. We shall come back in sect. 4 to the rational coefficients multiplying the expectation values on the *r.h.s.* of eqs. (2.17) and (2.19). For the time being, we finalize our argument by inserting these expressions into eq. (2.16). A little algebra yields

$$\frac{\alpha^2}{4\rho^2} \left\{ \sum_{k=1}^v \frac{\text{var}(X_k^2 | X \in \mathcal{B}_v(\rho))}{\lambda_k^2} - 2v + \sum_{j \neq k} \frac{\text{cov}(X_j^2, X_k^2 | X \in \mathcal{B}_v(\rho))}{\lambda_j \lambda_k} \right\} \leq 0. \quad (2.20)$$

Eq. (2.20) describes the log-concavity of α in terms of conditional expectations. Did we not know about Prékopa's Theorem, we could regard it as a result of eqs. (1.2) and (1.3). Unfortunately, the converse does not hold: it is not possible to infer eqs. (1.2) and (1.3) from eq. (2.20), as contributions along different directions could compensate while keeping the *l.h.s.* negative. Nevertheless, if eqs. (1.2) and (1.3) were simultaneously violated for all indices, $\alpha(\rho; \lambda)$ could not be logarithmic concave at all. Therefore, eq. (2.20) tells us that at least some of the correlation inequalities must hold. To conclude, a full proof of eq. (1.2) cannot follow from the property of log-concavity, so we need to look elsewhere.

3 Variance reduction in the regime of strong truncation

For the sake of conciseness, throughout this section we denote conditional expectations by $\mathbb{E}[\cdot]$ instead of $\mathbb{E}[\cdot | X \in \mathcal{B}_v(\rho)]$. Our starting point consists in regarding eq. (1.2) as an upper bound to $\mathbb{E}[X_n^4]$. This suggests to consider the wider inequality chain

$$\mathbb{E}[X_n^4] \leq \mathbb{E}[X_n^2] (2\lambda_n + \mathbb{E}[X_n^2]) \leq \lambda_n (2\lambda_n + \mathbb{E}[X_n^2]) \leq 3\lambda_n^2. \quad (3.1)$$

The leftmost bound is in fact a recast of eq. (1.2). If for a moment we give it for granted, the second and third ones follow as an immediate consequence of $\mathbb{E}[X_n^2] \leq \lambda_n$. Although our final target is just represented by eq. (1.2), it makes sense to first consider the two rightmost bounds: if they turn out to be violated, eq. (1.2) cannot be correct. The loosest one is once more the trivial inequality $\mathbb{E}[X_n^4] \leq 3\lambda_n^2$, which we have already established. By contrast, the inequality

$$\mathbb{E}[X_n^4] \leq \lambda_n (2\lambda_n + \mathbb{E}[X_n^2]) \quad (3.2)$$

is less obvious. In sect. 3.1 we prove it. Our argument is based on straightforward algebraic manipulations of the Gaussian integrals over $\mathcal{B}_v(\rho)$. We include it in the present note for a twofold reason: on the one hand it gives a feeling of the optimality of eq. (1.2), on the other it represents the only general result we have, valid across the whole parameter space.

3.1 A loose yet general bound to $\mathbb{E}[X_n^4 | X \in \mathcal{B}_v(\rho)]$

In order to prove eq. (3.2), we use a standard trick, consisting in a rescaling of λ_n by an external parameter τ , so as to obtain the moments of X_n by differentiation of α in τ . More precisely, we introduce the function

$$\mathcal{H}(\tau) = \frac{1}{\sqrt{\tau}} \alpha \left(\rho; \left\{ \lambda_1, \dots, \frac{\lambda_n}{\tau}, \dots, \lambda_v \right\} \right) = \frac{1}{\sqrt{\tau}} \int_{\mathcal{B}_v(\rho)} d^v x \, \delta(x_n; \lambda_n/\tau) \prod_{m \neq n} \delta(x_m, \lambda_m), \quad (3.3)$$

whose dependence upon ρ and λ we leave implicit. Differentiating $\mathcal{H}(\tau)$ under the integral sign yields

$$\mathbb{E}[X_n^{2k}] = (-1)^k \frac{2^k \lambda_n^k}{\alpha} \frac{\partial^k \mathcal{H}}{\partial \tau^k} \Big|_{\tau=1}, \quad k = 0, 1, 2, \dots \quad (3.4)$$

At the same time, derivatives of $\mathcal{H}(\tau)$ can be taken via the chain rule of differentiation, which allows us to express them as algebraic combinations of α and its derivatives in λ_n . For instance, with regard to the second and fourth moments, we find

$$\frac{\partial \mathcal{H}}{\partial \tau} = -\frac{1}{2\tau^{3/2}} (\alpha + 2\lambda_n \partial_n \alpha), \quad (3.5)$$

$$\frac{\partial^2 \mathcal{H}}{\partial \tau^2} = \frac{1}{4\tau^{5/2}} (3\alpha + 8\lambda_n \partial_n \alpha + 4\lambda_n^2 \partial_n^2 \alpha). \quad (3.6)$$

Consider first the lowest order derivative. By inserting eq. (3.5) into eq. (3.4) evaluated at $k = 1$, we obtain $\mathbb{E}[X_n^2] = \lambda_n [1 + 2(\lambda_n/\alpha) \partial_n \alpha]$. Comparing this with $\mathbb{E}[X_n^2] = \lambda_n (\alpha_n/\alpha)$ yields

$$\alpha_n = \alpha + 2\lambda_n \partial_n \alpha. \quad (3.7)$$

Eq. (3.7) coincides with eq. (2.8) evaluated at $n_1 = \dots = n_v = 0$. Owing to property $p_2)$ of sect. 2, we infer $\alpha_n \leq \alpha$ and thus we find again $\mathbb{E}[X_n^2] = \lambda_n (\alpha_n/\alpha) \leq \lambda_n$. Consider then the fourth moment. If we insert eq. (3.6) into eq. (3.4) evaluated at $k = 2$, and then make use of eq. (3.7), we easily arrive at

$$\mathbb{E}[X_n^4] = 4\lambda_n \mathbb{E}[X_n^2] - \lambda_n^2 + 4\lambda_n^4 \frac{\partial_n^2 \alpha}{\alpha}. \quad (3.8)$$

In order to estimate $\partial_n^2 \alpha$, we differentiate both sides of eq. (3.7) with respect to λ_n . We then invoke again property p_2) of sect. 2, thus obtaining

$$\partial_n^2 \alpha = \frac{1}{2\lambda_n} (\partial_n \alpha_n - 3\partial_n \alpha) \leq -\frac{3}{2\lambda_n} \partial_n \alpha = -\frac{3}{4\lambda_n^2} (\alpha_n - \alpha) = -\frac{3\alpha}{4\lambda_n^3} \{\mathbb{E}[X_n^2] - \lambda_n\}. \quad (3.9)$$

This estimate is sufficient to prove eq. (3.2).

3.2 First argument in favour of eq. (1.2)

In the regime of strong truncation, eq. (1.2) can be inferred from Hölder's inequality. We recall that if $p, q > 1$ are two numbers satisfying $1/p + 1/q = 1$ and X, Y are stochastic variables on a given probability space, then $\mathbb{E}[|XY|] \leq (\mathbb{E}[|X|^p])^{1/p} (\mathbb{E}[|Y|^q])^{1/q}$. In our case, we have

$$\begin{aligned} \text{var}(X_n^2) &= \mathbb{E} \left[(X_n^2 - \mathbb{E}[X_n^2])^2 \right] = \mathbb{E}[X_n^4] - \mathbb{E}[X_n^2]^2 = \mathbb{E}[X_n^4 - \mathbb{E}[X_n^2]^2] \\ &= \mathbb{E} \left[(X_n^2 - \mathbb{E}[X_n^2]) (X_n^2 + \mathbb{E}[X_n^2]) \right] \leq \mathbb{E} \left[|(X_n^2 - \mathbb{E}[X_n^2])| (X_n^2 + \mathbb{E}[X_n^2]) \right] \\ &\leq \left\{ \mathbb{E} \left[|X_n^2 - \mathbb{E}[X_n^2]|^p \right] \right\}^{1/p} \left\{ \mathbb{E} \left[(X_n^2 + \mathbb{E}[X_n^2])^q \right] \right\}^{1/q}. \end{aligned} \quad (3.10)$$

The latter inequality holds true for any finite choice of p, q , provided their reciprocals sum to one. Accordingly, it holds as well in the joint limit $q \rightarrow 1^+$, $p = q/(q-1) \rightarrow \infty$, where it amounts to

$$\text{var}(X_n^2) \leq 2h \mathbb{E}[X_n^2], \quad (3.11)$$

with

$$h \equiv \lim_{p \rightarrow \infty} \left\{ \mathbb{E} \left[|X_n^2 - \mathbb{E}[X_n^2]|^p \right] \right\}^{1/p} = \text{ess sup} (|X_n^2 - \mathbb{E}[X_n^2]|). \quad (3.12)$$

Recall that the essential supremum of a real-valued function f is defined by $\text{ess sup } f \equiv \inf\{a \in \mathbb{R} : \mu(\{x : f(x) > a\}) = 0\}$. In particular, the measure μ which is understood in eq. (3.12) is the marginal probability measure of X_n , *i.e.*

$$d\mu(x_n) = \frac{\alpha^{(v-1)}(\rho - x_n^2; \lambda_{(n)}) \delta(x_n, \lambda_n)}{\alpha^{(v)}(\rho; \lambda)} dx_n. \quad (3.13)$$

Owing to the modulating factor $\alpha^{(v-1)}(\rho - x_n^2; \lambda_{(n)})$, μ is neither Gaussian nor log-concave (in x_n). In sect. 4 we shall say more about eq. (3.13) and the factorization of its numerator into functions of resp. λ_n and $\lambda_{(n)}$. For the time being, we observe that μ has support in the interval $(-\sqrt{\rho}, +\sqrt{\rho})$. Depending on how $\mathbb{E}[X_n^2]$ compares with $\rho/2$, h might assume one of the values $h_1 = \rho - \mathbb{E}[X_n^2]$ or $h_2 = \mathbb{E}[X_n^2]$, as qualitatively represented in Fig. 2. As far as we are concerned, we do not need to establish which among Fig. 2a and Fig. 2b provides the correct qualitative behavior for $x_n \mapsto |x_n^2 - \mathbb{E}[X_n^2]|$: numerical computations suggest that Fig. 2b is not realized for any choice of n, ρ and λ , yet this information is irrelevant for what follows. More precisely, we distinguish three cases:

- i) $\rho \leq \lambda_n$ (*strong truncation*): in this case $h \leq \lambda_n$. Indeed, since $\mathbb{E}[X_n^2] \leq \lambda_n$, both h_1 and h_2 lie below λ_n . In this region, we have no analytic argument in favour of Fig. 2a or Fig. 2b.
- ii) $\rho > 2\lambda_n$ (*weak truncation*): in this case $h > \lambda_n$. Indeed, again from $\mathbb{E}[X_n^2] \leq \lambda_n$, we deduce $\rho - \mathbb{E}[X_n^2] \geq \rho - \lambda_n > \lambda_n \geq \mathbb{E}[X_n^2]$. Here, the correct profile of $|x_n^2 - \mathbb{E}[X_n^2]|$ is certainly the one depicted in Fig. 2a.
- iii) $\lambda_n < \rho < 2\lambda_n$: in this case it is difficult to conclude anything about h , except that by continuity there exists a value $\lambda_n < \rho_*(\lambda_{(n)}) < 2\lambda_n$, possibly depending on $\lambda_{(n)}$, such that $h \leq \lambda_n \Leftrightarrow \rho \leq \rho_*$.

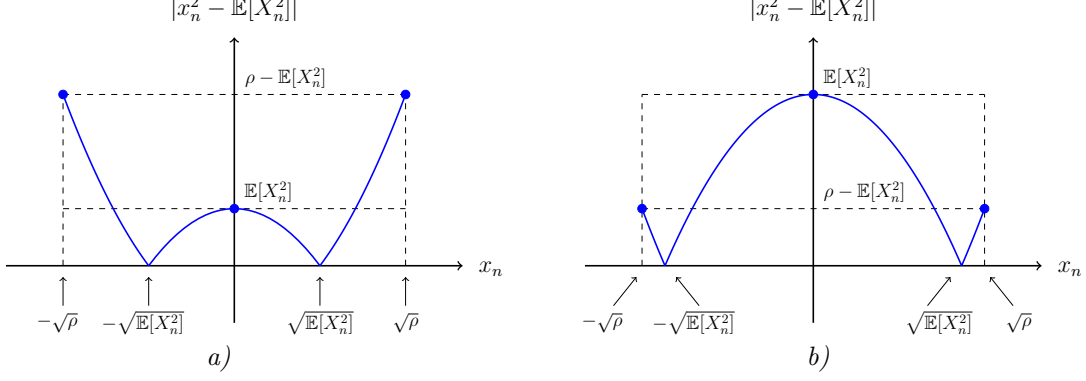


Fig. 2 – Qualitative behavior of the function $x_n \mapsto |x_n^2 - \mathbb{E}[X_n^2]|$ within the support $(-\sqrt{\rho}, +\sqrt{\rho})$. Note that, depending on ρ and $\mathbb{E}[X_n^2]$, the function might have one or two maxima. Specifically, plot *a*) represents the function if $\rho - \mathbb{E}[X_n^2] > \mathbb{E}[X_n^2]$, while plot *b*) applies if $\rho - \mathbb{E}[X_n^2] < \mathbb{E}[X_n^2]$.

To conclude, the estimate obtained from Hölder's inequality is certainly as strict as needed for eq. (1.2) to hold true only in case of strong truncation, *i.e.* for $\rho \leq \lambda_n$. In addition, there is a crossover region where the same estimate might be sufficiently strict, while it becomes definitely too loose in the region of weak truncation.

3.3 Second argument in favour of eq. (1.2)

In order to extend the above proof to the region $0 < \rho \leq 2\lambda_n$, we work on the integral representations of $\mathbb{E}[X_n^4] = \lambda_n^2(\alpha_{nn}/\alpha)$ and $\mathbb{E}[X_n^2] = \lambda_n(\alpha_n/\alpha)$. In terms of these, Δ_n reads

$$\Delta_n = \frac{\lambda_n^2}{\rho^2} \left[\frac{\alpha_{nn}}{\alpha} - \left(\frac{\alpha_n}{\alpha} \right)^2 - 2 \frac{\alpha_n}{\alpha} \right]. \quad (3.14)$$

Now, we observe that independently of v , $\alpha_{n:k}$ is bounded from above by $(\rho/\lambda_n)^{k-p} \alpha_{n:p}$ for any $p < k$. Indeed, since $x \in \mathcal{B}_v(\rho) \Rightarrow -\sqrt{\rho} < x_n < \sqrt{\rho}$, we have

$$\alpha_{n:k} = \frac{\rho^k}{\lambda_n^k} \int_{\mathcal{B}_v(\rho)} d^v x \frac{x_n^{2k}}{\rho^k} \prod_{m=1}^v \delta(x_m, \lambda_m) \leq \frac{\rho^k}{\lambda_n^k} \int_{\mathcal{B}_v(\rho)} d^v x \frac{x_n^{2p}}{\rho^p} \prod_{m=1}^v \delta(x_m, \lambda_m) = \frac{\rho^{k-p}}{\lambda_n^{k-p}} \alpha_{n:p}. \quad (3.15)$$

Thus, we immediately obtain

$$\Delta_n \leq \frac{\lambda_n^2}{\rho^2} \left[\left(\frac{\rho}{\lambda_n} - 2 \right) \frac{\alpha_n}{\alpha} - \left(\frac{\alpha_n}{\alpha} \right)^2 \right] \leq 0, \quad \text{if } \rho \leq 2\lambda_n. \quad (3.16)$$

This conclusion is somewhat conservative, as indeed $\Delta_n \leq 0 \Leftrightarrow \rho \leq \rho_*$, being ρ_* implicitly defined by the non-linear equation $\rho_* = 2\{\lambda_n + \mathbb{E}[X_n^2 | X \in \mathcal{B}_v(\rho_*)]\}$. By continuity, the latter is certainly fulfilled by some $2\lambda_n < \rho_* \leq 4\lambda_n$. The argument presented here does not apply for $\rho > 4\lambda_n$.

4 Weak truncation expansion

In order to study the variance of the square components of X in the regime of weak truncation, we need to develop an appropriate formalism. To start with, we observe that the constraint $X \in \mathcal{B}_v(\rho)$ becomes increasingly unrestrictive as $\rho \rightarrow \infty$. As a consequence, we have the asymptotic factorization

$$\alpha_{1:k_1 \dots v:k_v}^{(v)}(\rho; \lambda) \stackrel{\rho \gg \max_j \{\lambda_j\}}{\sim} \prod_{j=1}^v \alpha_{j:k_j}^{(1)}(\rho; \lambda_j). \quad (4.1)$$

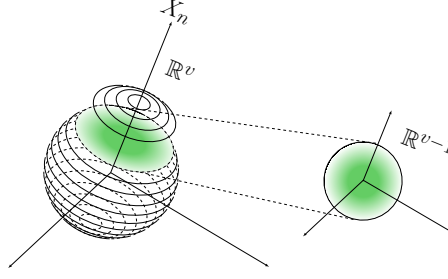


Fig. 3 – Slicing $\mathcal{B}_v(\rho)$ orthogonally to the n^{th} coordinate axis \hat{X}_n amounts to representing it as an uncountable union of Euclidean balls, living in the $(v-1)$ -dimensional subspace orthogonal to \hat{X}_n , with square radius varying from 0 to ρ .

The larger is ρ , the less is the error made in approximating $\alpha_{1:k_1 \dots v:k_v}$ by its factorized counterpart. We aim at characterizing the corrections to eq. (4.1) when ρ is large yet finite. Actually, we are not interested in a complete factorization of the Gaussian integrals: if $\rho \gg \lambda_n$ just for some $1 \leq n \leq v$, we look at the partial factorization occurring along the n^{th} direction. Note that: *i*) in the regime of weak truncation, every rational combination of Gaussian integrals — such as Δ_n — is led by its factorized counterpart; as we shall see, the latter is subject to relevant simplifications in case of ratios of integrals; *ii*) 1-dimensional integrals cannot be further simplified, as they amount to lower incomplete gamma functions,

$$\alpha_{n:k}^{(1)}(\rho; \lambda_n) = \frac{2^k}{\sqrt{\pi}} \gamma\left(k + \frac{1}{2}, \frac{\rho}{2\lambda_n}\right), \quad \gamma(s, x) = \int_0^x dt t^{s-1} e^{-t}. \quad (4.2)$$

4.1 Expansion of Gaussian integrals

In order to present the idea, we first focus on α . If $\rho \gg \lambda_n$ for some $1 \leq n \leq v$, we slice the integration domain orthogonally to the n^{th} direction, as depicted in Fig. 3. From a geometrical point of view, this corresponds to representing $\mathcal{B}_v(\rho)$ as an uncountable union of $(v-1)$ -dimensional Euclidean balls, *i.e.*

$$\mathcal{B}_v(\rho) = \bigcup_{x_n \in (-\sqrt{\rho}, +\sqrt{\rho})} \{y \in \mathbb{R}^v : y_n = x_n, y_{(n)} \in \mathcal{B}_{v-1}(\rho - x_n^2)\}. \quad (4.3)$$

Such technique has been first considered by Ruben [4] with the aim of obtaining an integral recurrence relationship on the dimensionality of α . Interpreting the integration domain in terms of eq. (4.3) indeed yields

$$\alpha^{(v)}(\rho; \lambda) = \int_{-\sqrt{\rho}}^{+\sqrt{\rho}} dx_n \delta(x_n, \lambda_n) \alpha^{(v-1)}(\rho - x_n^2; \lambda_{(n)}). \quad (4.4)$$

Since $\alpha(\rho - x_n^2; \lambda_{(n)})$ is a smooth function of its first argument $\rho - x_n^2$, we propose to expand it in Taylor series around $x_n^2 = 0^+$,

$$\begin{aligned} \alpha^{(v-1)}(\rho - x_n^2; \lambda_{(n)}) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \lambda_n^k \left(\frac{x_n^2}{\lambda_n}\right)^k \partial_{\rho}^k \alpha^{(v-1)}(\rho; \lambda_{(n)}) \\ &= \alpha^{(v-1)}(\rho; \lambda_{(n)}) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\lambda_n}{\rho}\right)^k \left(\frac{x_n^2}{\lambda_n}\right)^k \eta_k^{(v-1)}(\rho; \lambda_{(k)}), \end{aligned} \quad (4.5)$$

with the functions η_k defined by

$$\eta_k^{(v)}(\rho; \lambda) = \begin{cases} 1, & k = 0, \\ [\alpha^{(v)}(\rho; \lambda)]^{-1} \rho^k \partial_\rho^k \alpha^{(v)}(\rho; \lambda), & k \geq 1. \end{cases} \quad (4.6)$$

When inserted into eq. (4.4), the Taylor series turns into a weak truncation expansion of α , namely

$$\begin{aligned} \alpha^{(v)}(\rho; \lambda) &= \alpha^{(v-1)}(\rho; \lambda_{(n)}) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\lambda_n}{\rho} \right)^k \alpha_{n:k}^{(1)}(\rho; \lambda_n) \eta_k^{(v-1)}(\rho; \lambda_{(n)}) \\ &= \alpha^{(1)}(\rho; \lambda_n) \alpha^{(v-1)}(\rho; \lambda_{(n)}) - \frac{\lambda_n}{\rho} \alpha_n^{(1)}(\rho; \lambda_n) \alpha^{(v-1)}(\rho; \lambda_{(n)}) \eta_1^{(v-1)}(\rho; \lambda_{(n)}) \\ &\quad + \frac{1}{2} \frac{\lambda_n^2}{\rho^2} \alpha_{nn}^{(1)}(\rho; \lambda_n) \alpha^{(v-1)}(\rho; \lambda_{(n)}) \eta_2^{(v-1)}(\rho; \lambda_{(n)}) + \mathcal{O}\left(\frac{\lambda_n^3}{\rho^3}\right). \end{aligned} \quad (4.7)$$

Although a complete factorization into functions of λ_n and $\lambda_{(n)}$ is not exactly realized at finite ρ , we see that it occurs at each order of the expansion. We warn that eq. (4.7) has been obtained upon bringing an infinite sum under an integral sign. Such exchange of limits is delicate, so it is not *a priori* evident whether the resulting expansion converges or approximates its target just as an asymptotic series. We shall come back to this point later on. We also stress that, while power counting is performed by factors of $(\lambda_n/\rho)^k$, additional powers and exponentially small terms in ρ are still hidden within the coefficient functions² $\alpha_{n:k}^{(1)}$ and $\eta_k^{(v-1)}$.

To simplify the notation, in the sequel we drop all function arguments, whenever this does not generate confusion. Thus, we shorten eq. (4.7) to

$$\alpha^{(v)} = \alpha^{(1)} \alpha^{(v-1)} - \frac{\lambda_n}{\rho} \alpha_n^{(1)} \alpha^{(v-1)} \eta_1^{(v-1)} + \frac{1}{2} \frac{\lambda_n^2}{\rho^2} \alpha_{nn}^{(1)} \alpha^{(v-1)} \eta_2^{(v-1)} + \mathcal{O}\left(\frac{\lambda_n^3}{\rho^3}\right). \quad (4.8)$$

The same technique can be straightforwardly applied to $\alpha_{n:p}$. For instance, we have for $p = 1, 2, \dots$

$$\alpha_n^{(v)} = \alpha_{n:1}^{(1)} \alpha^{(v-1)} - \frac{\lambda_n}{\rho} \alpha_{n:2}^{(1)} \alpha^{(v-1)} \eta_1^{(v-1)} + \frac{1}{2} \frac{\lambda_n^2}{\rho^2} \alpha_{n:3}^{(1)} \alpha^{(v-1)} \eta_2^{(v-1)} + \mathcal{O}\left(\frac{\lambda_n^3}{\rho^3}\right), \quad (4.9)$$

$$\alpha_{nn}^{(v)} = \alpha_{n:2}^{(1)} \alpha^{(v-1)} - \frac{\lambda_n}{\rho} \alpha_{n:3}^{(1)} \alpha^{(v-1)} \eta_1^{(v-1)} + \frac{1}{2} \frac{\lambda_n^2}{\rho^2} \alpha_{n:4}^{(1)} \alpha^{(v-1)} \eta_2^{(v-1)} + \mathcal{O}\left(\frac{\lambda_n^3}{\rho^3}\right), \quad (4.10)$$

\vdots

Since the above expansions are all based on eq. (4.5), the coefficient functions η_k are the same independently of p . By contrast, the multiplicity of the index n of the 1-dimensional integrals contributing to each order is shifted forward as p increases.

Now, when it comes to expanding Gaussian integrals with more than one index, the above procedure is carried out in a slightly different way. For instance, in order to expand α_{nm} we need to take into account factors of x_n^2 and x_m^2 under the integral sign. Accordingly, we slice $\mathcal{B}_v(\rho)$ subsequently along the n^{th} and m^{th} directions under the assumption $\rho \gg \max\{\lambda_n, \lambda_m\}$. The analogous of eq. (4.4) reads

$$\alpha_{nm}^{(v)}(\rho; \lambda) = \int_{-\sqrt{\rho}}^{+\sqrt{\rho}} dx_n \frac{x_n^2}{\lambda_n} \delta(x_n, \lambda_n) \int_{-\sqrt{\rho}}^{+\sqrt{\rho}} dx_m \frac{x_m^2}{\lambda_m} \delta(x_m, \lambda_m) \alpha^{(v-2)}(\rho - x_n^2 - x_m^2; \lambda_{(nm)}), \quad (4.11)$$

²The reader will observe that the coefficient function $\alpha^{(v-1)}$ showing up in each term of the expansion is totally useless, as it simplifies with the one attached to $\eta_k^{(v-1)}$. Such redundancy is real, yet it turns useful when ratios of Gaussian integrals are considered, as we shall see in sects. 4.2 and 5.

with $\lambda_{(nm)} \equiv \{\lambda_k\}_{k \neq n, m}$. Again, we expand $\alpha(\rho - x_n^2 - x_m^2; \lambda_{(nm)})$ in Taylor series around the point $x_n^2 + x_m^2 = 0^+$, thus obtaining

$$\begin{aligned}
\alpha^{(v-2)}(\rho - x_n^2 - x_m^2; \lambda_{(nm)}) &= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} (x_n^2 + x_m^2)^j \partial_{\rho}^j \alpha^{(v-2)}(\rho; \lambda_{(nm)}) \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=0}^j \binom{j}{k} x_n^{2k} x_m^{2(j-k)} \partial_{\rho}^j \alpha^{(v-2)}(\rho; \lambda_{(nm)}) \\
&= \alpha^{(v-2)}(\rho; \lambda_{(nm)}) \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \sum_{k=0}^j \binom{j}{k} \left(\frac{\lambda_n}{\rho}\right)^k \left(\frac{\lambda_m}{\rho}\right)^{j-k} \\
&\quad \cdot \left(\frac{x_n^2}{\lambda_n}\right)^k \left(\frac{x_m^2}{\lambda_m}\right)^{j-k} \eta_j^{(v-2)}(\rho; \lambda_{(nm)}). \tag{4.12}
\end{aligned}$$

Here we have also used Newton's binomial formula to express each term of the series in products of powers of x_n^2 and x_m^2 . Inserting this expression back into eq. (4.11) yields

$$\alpha_{nm}^{(v)} = \alpha_n^{(1)} \alpha_m^{(1)} \alpha^{(v-2)} - \frac{\lambda_n}{\rho} \alpha_{n:2}^{(1)} \alpha_m^{(1)} \alpha^{(v-2)} \eta_1^{(v-2)} - \frac{\lambda_m}{\rho} \alpha_n^{(1)} \alpha_{m:2}^{(1)} \alpha^{(v-2)} \eta_1^{(v-2)} + O\left(\frac{\lambda^2}{\rho^2}\right). \tag{4.13}$$

Eq. (4.12) can be also used to obtain alternative expansions of $\alpha_{n:p}$ if $\rho \gg \max\{\lambda_n, \lambda_m\}$ for some m , *e.g.*

$$\alpha^{(v)} = \alpha^{(1)} \alpha^{(1)} \alpha^{(v-2)} - \frac{\lambda_n}{\rho} \alpha_{n:1}^{(1)} \alpha^{(1)} \alpha^{(v-2)} \eta_1^{(v-2)} - \frac{\lambda_m}{\rho} \alpha_{m:1}^{(1)} \alpha^{(1)} \alpha^{(v-2)} \eta_1^{(v-2)} + O\left(\frac{\lambda^2}{\rho^2}\right), \tag{4.14}$$

$$\alpha_n^{(v)} = \alpha_{n:1}^{(1)} \alpha^{(1)} \alpha^{(v-2)} - \frac{\lambda_n}{\rho} \alpha_{n:2}^{(1)} \alpha^{(1)} \alpha^{(v-2)} \eta_1^{(v-2)} - \frac{\lambda_m}{\rho} \alpha_{n:1}^{(1)} \alpha_{m:1}^{(1)} \alpha^{(v-2)} \eta_1^{(v-2)} + O\left(\frac{\lambda^2}{\rho^2}\right), \tag{4.15}$$

⋮

4.2 Exercise: relative amplitude of variances and covariances

We make use of the above expansions to qualitatively compare the correlations X_n^2 has with itself and the other square components of X in the regime of weak truncation. This is rather instructive, because it shows how analytic cancellations occur in the proposed formalism. In addition, the exercise inspires the following unproved

Conjecture 4.1. *If $X \sim \mathcal{N}_v(0, \Lambda)$ with $\Lambda = \text{diag}(\lambda)$, the covariance matrix of the vector $\{X_n^2\}_{n=1}^v$ conditioned to $\mathcal{B}_v(\rho)$ is diagonally dominant, i.e.*

$$\text{var}(X_n^2 | X \in \mathcal{B}_v(\rho)) \geq \sum_{m \neq n} |\text{cov}(X_n^2, X_m^2 | X \in \mathcal{B}_v(\rho))|, \quad \rho \in \mathbb{R}_+. \tag{4.16}$$

□

Eq. (4.16) is supported with no exceptions by extensive numerical tests. As observed in ref. [5], it entails the inequality $v^{-1} \sum_{k=1}^v \mu_k \leq \rho/(v+2)$, where $\mu = \{\mu_k\}_{k=1}^v$ denotes the eigenvalue spectrum of the covariance matrix $\mathfrak{S}_{\mathcal{B}}$.

In terms of Gaussian integrals the scale invariant observables we focus on are

$$\Gamma_{nn}^{(v)} \equiv \frac{1}{\rho^2} \text{var}(X_n^2 | X \in \mathcal{B}_v(\rho)) = \frac{\lambda_n^2}{\rho^2} \left[\frac{\alpha_{nn}^{(v)}}{\alpha^{(v)}} - \left(\frac{\alpha_n^{(v)}}{\alpha^{(v)}} \right)^2 \right], \tag{4.17}$$

$$\Gamma_{nm}^{(v)} \equiv \frac{1}{\rho^2} \text{cov}(X_n^2, X_m^2 | X \in \mathcal{B}_v(\rho)) = \frac{\lambda_n}{\rho} \frac{\lambda_m}{\rho} \left[\frac{\alpha_{nm}^{(v)}}{\alpha^{(v)}} - \frac{\alpha_n^{(v)}}{\alpha^{(v)}} \frac{\alpha_m^{(v)}}{\alpha^{(v)}} \right], \quad n \neq m. \tag{4.18}$$

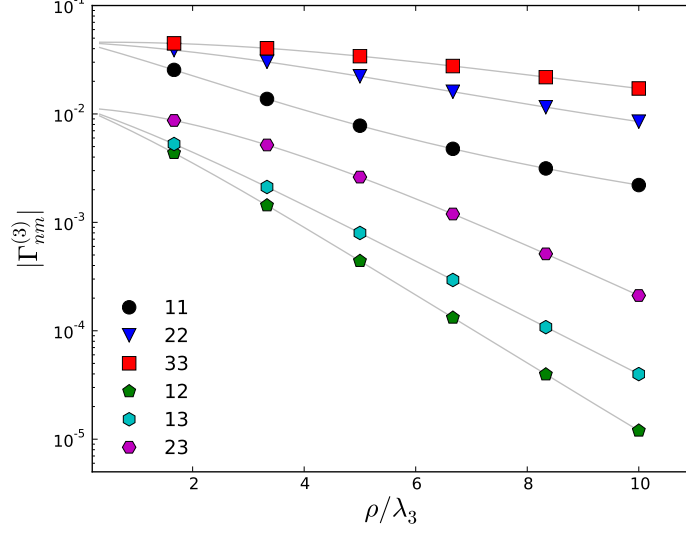


Fig. 4 – A plot of $|\Gamma_{nm}^{(3)}|$ vs. ρ/λ_3 at $\{\lambda_1, \lambda_2, \lambda_3\} = \{1, 2, 3\}$. Solid curves correspond to computed values: markers have been overimposed just to make the legend clear.

For illustrative purposes, we show in Fig. 4 a plot of $|\Gamma_{nm}^{(v)}|$ vs. ρ/λ_3 at $v = 3$ corresponding to the choice $\{\lambda_1, \lambda_2, \lambda_3\} = \{1, 2, 3\}$. Both Γ_{nn} and Γ_{nm} vanish as $\rho \rightarrow \infty$, yet the former vanishes as $1/\rho^2$ due to the chosen normalization, whereas the latter is exponentially damped.

We use eqs. (4.8)–(4.10) to work out the expansion of Γ_{nn} and eqs. (4.13)–(4.15) for Γ_{nm} . In both cases, in order to expand α^{-1} we rely on the Taylor formula $(1 - x)^{-1} = 1 + x + x^2 + O(x^3)$. Thus, with regard to Γ_{nn} we have

$$\frac{\alpha_{nn}^{(v)}}{\alpha^{(v)}} = \frac{\alpha_{nn}^{(1)}}{\alpha^{(1)}} - \frac{\lambda_n}{\rho} \left[\frac{\alpha_{n:3}^{(1)}}{\alpha^{(1)}} - \frac{\alpha_{n:2}^{(1)}}{\alpha^{(1)}} \frac{\alpha_n^{(1)}}{\alpha^{(1)}} \right] \eta_1^{(v-1)} + O\left(\frac{\lambda_n^2}{\rho^2}\right), \quad (4.19)$$

$$\left(\frac{\alpha_n^{(v)}}{\alpha^{(v)}}\right)^2 = \left(\frac{\alpha_n^{(1)}}{\alpha^{(1)}}\right)^2 - 2\frac{\lambda_n}{\rho} \left[\frac{\alpha_{n:2}^{(1)}}{\alpha^{(1)}} \frac{\alpha_n^{(1)}}{\alpha^{(1)}} - \left(\frac{\alpha_n^{(1)}}{\alpha^{(1)}}\right)^3 \right] \eta_1^{(v-1)} + O\left(\frac{\lambda_n^2}{\rho^2}\right), \quad (4.20)$$

whence we obtain

$$\Gamma_{nn}^{(v)} = \Gamma_{nn}^{(1)} - \frac{\lambda_n^3}{\rho^3} \left[\frac{\alpha_{n:3}^{(1)}}{\alpha^{(1)}} - 3\frac{\alpha_{n:2}^{(1)}}{\alpha^{(1)}} \frac{\alpha_n^{(1)}}{\alpha^{(1)}} + 2\left(\frac{\alpha_n^{(1)}}{\alpha^{(1)}}\right)^3 \right] \eta_1^{(v-1)} + O\left(\frac{\lambda_n^2}{\rho^2}\right). \quad (4.21)$$

We see that the leading term of Γ_{nn} coincides with its 1-dimensional counterpart. Since $\lim_{\rho \rightarrow \infty} \alpha_{n:k}^{(1)} = (2k-1)!!$, we have $\lim_{\rho \rightarrow \infty} (\rho^2/\lambda_n^2) \Gamma_{nn}^{(1)} = 2$, and thus we find again

$$\Gamma_{nn}^{(v)} \underset{\rho \gg \lambda_n}{\sim} \frac{\lambda_n^2}{\rho^2} \left\{ 2 + O\left(\frac{\lambda_n}{\rho}\right) \right\}, \quad (4.22)$$

apart from exponentially small terms in ρ . Analogously, we have

$$\begin{aligned} \frac{\alpha_{nm}^{(v)}}{\alpha^{(v)}} &= \frac{\alpha_n^{(1)}}{\alpha^{(1)}} \frac{\alpha_m^{(1)}}{\alpha^{(1)}} - \frac{\lambda_n}{\rho} \left[\frac{\alpha_{n:2}^{(1)}}{\alpha^{(1)}} - \frac{\alpha_m^{(1)}}{\alpha^{(1)}} \left(\frac{\alpha_n^{(1)}}{\alpha^{(1)}}\right)^2 \right] \eta_1^{(v-2)} \\ &\quad - \frac{\lambda_m}{\rho} \left[\frac{\alpha_{m:2}^{(1)}}{\alpha^{(1)}} - \frac{\alpha_n^{(1)}}{\alpha^{(1)}} \left(\frac{\alpha_m^{(1)}}{\alpha^{(1)}}\right)^2 \right] \eta_1^{(v-2)} + O\left(\frac{\lambda^2}{\rho^2}\right), \end{aligned} \quad (4.23)$$

$$\begin{aligned} \frac{\alpha_n^{(v)} \alpha_m^{(v)}}{\alpha^{(v)}} &= \frac{\alpha_n^{(1)} \alpha_m^{(1)}}{\alpha^{(1)}} - \frac{\lambda_n}{\rho} \left[\frac{\alpha_{n:2}^{(1)}}{\alpha^{(1)}} - \frac{\alpha_m^{(1)}}{\alpha^{(1)}} \left(\frac{\alpha_n^{(1)}}{\alpha^{(1)}} \right)^2 \right] \eta_1^{(v-2)} \\ &\quad - \frac{\lambda_m}{\rho} \left[\frac{\alpha_{m:2}^{(1)}}{\alpha^{(1)}} - \frac{\alpha_n^{(1)}}{\alpha^{(1)}} \left(\frac{\alpha_m^{(1)}}{\alpha^{(1)}} \right)^2 \right] \eta_1^{(v-2)} + O\left(\frac{\lambda^2}{\rho^2}\right). \end{aligned} \quad (4.24)$$

When eqs. (4.23)–(4.24) are put into eq. (4.18), an exact cancellation occurs separately among the $O(1)$ – and $O(\lambda/\rho)$ –terms, so we are left with

$$\Gamma_{nm}^{(v)} \quad \rho \gg \underbrace{\max\{\lambda_n, \lambda_m\}} \quad \frac{\lambda_n}{\rho} \frac{\lambda_m}{\rho} O\left(\frac{\lambda^2}{\rho^2}\right). \quad (4.25)$$

Eqs. (4.22) and (4.25) reflect the behavior observed in Fig. 4.

4.3 Asymptotic vanishing of η_k from combinatorial arguments

In order to make the weak truncation expansion effective, we need to characterize the coefficient functions η_k and provide an algorithmic recipe for their computation. A trivial property, *i.e.* the vanishing of η_k as $\rho \rightarrow \infty$, can be proved from purely combinatorial arguments based on eq. (2.3). This kind of proof nicely follows as a simple application of the scaling eq. (2.1), yet it gives no clue to the vanishing rate of η_k . We begin with the following

Lemma 4.1. *Given a set of $m \geq 0$ distinct indices $\{n_1, \dots, n_m\}$ and a corresponding set of strictly positive multiplicities $\{k_1, \dots, k_m\}$, we have*

$$\lim_{\rho \rightarrow \infty} \rho \partial_\rho \alpha_{n_1:k_1 \dots n_m:k_m}^{(v)}(\rho; \lambda) = 0. \quad (4.26)$$

Proof. From eq. (2.3), it follows

$$\lim_{\rho \rightarrow \infty} \rho \partial_\rho \alpha_{n_1:k_1 \dots n_m:k_m}^{(v)} = -\frac{1}{2} \sum_{n=1}^v \lim_{\rho \rightarrow \infty} \alpha_{n_1:k_1 \dots n_m:k_m n}^{(v)} + \frac{1}{2}(v+2k) \lim_{\rho \rightarrow \infty} \alpha_{n_1:k_1 \dots n_m:k_m}^{(v)}, \quad (4.27)$$

with $k = \sum_{j=1}^m k_j$. The sum index n in eq. (4.27) can either match one of the indices n_1, \dots, n_m or none of them. Moreover, we have

$$\lim_{\rho \rightarrow \infty} \alpha_{n_1:k_1 \dots n_m:k_m n}^{(v)} = \begin{cases} \lim_{\rho \rightarrow \infty} \alpha_{n_1:k_1 \dots n_m:k_m}^{(v)} & \text{if } n \neq n_j \ \forall j = 1, \dots, m; \\ (2k_j + 1) \lim_{\rho \rightarrow \infty} \alpha_{n_1:k_1 \dots n_m:k_m}^{(v)} & \text{if } n = n_j \text{ for some } j; \end{cases}, \quad (4.28)$$

as a consequence of the exact factorization of the Gaussian integrals as $\rho \rightarrow \infty$ and the standard formula $\mathbb{E}[z^{2n}] = (2n-1)!!$ ($n \geq 0$), valid for $z \sim \mathcal{N}(0, 1)$. Therefore,

$$\begin{aligned} \lim_{\rho \rightarrow \infty} \rho \partial_\rho \alpha_{n_1:k_1 \dots n_m:k_m}^{(v)} &= \lim_{\rho \rightarrow \infty} \alpha_{n_1:k_1 \dots n_m:k_m}^{(v)} \cdot \left[-\frac{v-m}{2} - \frac{1}{2} \sum_{j=1}^m (2k_j + 1) + \frac{1}{2}(v+2k) \right] \\ &= \lim_{\rho \rightarrow \infty} \alpha_{n_1:k_1 \dots n_m:k_m}^{(v)} \cdot \left[\frac{m+2k}{2} - \frac{m+2k}{2} \right] = 0. \end{aligned} \quad (4.29)$$

□

From Lemma 4.1 we easily derive

Proposition 4.1. *As $\rho \rightarrow \infty$ all the coefficient functions η_k vanish.*

Proof. Let us define $f_k = (\rho\partial_\rho)^k \alpha^{(v)}$ and $x_k = \sum_{n_1 \dots n_k=1}^v \alpha_{n_1 \dots n_k}^{(v)}$. We first prove by induction that f_k is a homogeneous linear function of x_0, \dots, x_k . From eq. (2.3), evaluated at $k_1 = \dots = k_n = 0$, we have indeed

$$\rho\partial_\rho \alpha^{(v)} = \frac{v}{2} \alpha^{(v)} - \frac{1}{2} \sum_{k=1}^v \alpha_k^{(v)} = \frac{v}{2} x_0 - \frac{1}{2} x_1 = f_1(x_0, x_1). \quad (4.30)$$

Now, suppose that f_{k-1} is a homogeneous linear function of x_0, \dots, x_{k-1} . Then,

$$\begin{aligned} f_k &= (\rho\partial_\rho)^k \alpha^{(v)} = (\rho\partial_\rho)(\rho\partial_\rho)^{k-1} \alpha^{(v)} = \rho\partial_\rho f_{k-1}(x_0, \dots, x_{k-1}) \\ &= f_{k-1}(\rho\partial_\rho x_0, \dots, \rho\partial_\rho x_{k-1}) = f_{k-1} \left(\rho\partial_\rho \alpha^{(v)}, \dots, \sum_{n_1 \dots n_{k-1}=1}^v \rho\partial_\rho \alpha_{n_1 \dots n_{k-1}}^{(v)} \right). \end{aligned} \quad (4.31)$$

The inductive step follows from eq. (2.3) and the assumed linearity of f_{k-1} . Hence, we have

$$\begin{aligned} \lim_{\rho \rightarrow \infty} (\rho\partial_\rho)^k \alpha^{(v)} &= \lim_{\rho \rightarrow \infty} f_k(x_0, \dots, x_k) \\ &= f_{k-1} \left(\lim_{\rho \rightarrow \infty} \rho\partial_\rho \alpha, \dots, \sum_{n_1, \dots, n_{k-1}} \lim_{\rho \rightarrow \infty} \rho\partial_\rho \alpha_{n_1, \dots, n_{k-1}}^{(v)} \right) = f_{k-1}(0, \dots, 0) = 0, \end{aligned} \quad (4.32)$$

where the last equality is again a consequence of the homogeneous linearity of f_{k-1} and the second-to-last one follows from Lemma 4.1. In addition, we known that (see for instance exercise 13, chap. 6 of ref. [6])

$$\rho^k \partial_\rho^k = \sum_{j=1}^k (-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix} (\rho\partial_\rho)^j, \quad (4.33)$$

with the symbols $\begin{bmatrix} k \\ j \end{bmatrix}$ denoting unsigned Stirling numbers of the first kind. Hence, we conclude

$$\lim_{\rho \rightarrow \infty} \eta_k^{(v)} = \lim_{\rho \rightarrow \infty} \frac{1}{\alpha^{(v)}} \sum_{j=1}^k (-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix} (\rho\partial_\rho)^j \alpha^{(v)} = \sum_{j=1}^k (-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix} \lim_{\rho \rightarrow \infty} (\rho\partial_\rho)^j \alpha^{(v)} = 0. \quad (4.34)$$

□

4.4 Gaussian representation of η_k

The above discussion suggests a convenient way to compute the coefficient functions. We have just seen that η_k is a linear combination of f_1, \dots, f_k . Moreover, $\forall j \geq 0$ f_j is a linear combination of x_0, \dots, x_j . We conclude that η_k itself can be represented as a linear combination of x_0, \dots, x_k . Since we know how to compute Gaussian integrals with controlled uncertainty, we have a complete recipe for η_k , provided we determine the coefficients of such linear combinations. To this aim, we concentrate first on the f_k 's. Eq. (4.30) gives the analytic expression of f_1 . By direct calculation we can also derive the expressions

$$f_2(x_0, x_1, x_2) = \frac{v^2}{4} x_0 - \frac{v+1}{2} x_1 + \frac{1}{4} x_2, \quad (4.35)$$

$$f_3(x_0, x_1, x_2, x_3) = \frac{v^3}{8} x_0 - \frac{3v^2+6v+4}{8} x_1 + \frac{3v+6}{8} x_2 - \frac{1}{8} x_3, \quad (4.36)$$

⋮

A generalization is provided by the following

Proposition 4.2. For $k \geq 1$, we have

$$f_k(x_0, \dots, x_k) = \sum_{\ell=0}^k d_{k\ell}(v) x_\ell, \quad (4.37)$$

where the coefficients $d_{k\ell}(v)$ are defined by

$$d_{k\ell}(v) = \begin{cases} \sum_{t=\ell}^k \frac{(-1)^\ell}{2^t} \phi_{t-\ell} \left\{ \begin{matrix} k \\ t \end{matrix} \right\} \binom{t}{\ell}, & k = 0, \dots, \ell, \\ 0, & \text{otherwise,} \end{cases} \quad (4.38)$$

$$\phi_k = \frac{v!!}{(v-2k)!!} = \begin{cases} 1 & k = 0 \\ (v-2k+2) \phi_{k-1} & k \geq 1 \end{cases}, \quad (4.39)$$

and the symbols $\left\{ \begin{matrix} k \\ t \end{matrix} \right\}$ denote Stirling numbers of the second kind.

Proof. The proof is by induction. We first note that $d_{10}(v) = v/2$ and $d_{11}(v) = -1/2$. Hence, for $k = 1$ eq. (4.37) agrees with eq. (4.30). Now, suppose that f_k is well represented by eq. (4.37) with $d_{k\ell}(v)$ and ϕ_k as in eqs. (4.38) and (4.39). Then, from eq. (2.3) it follows

$$\begin{aligned} f_{k+1} &= (\rho \partial_\rho)^{k+1} \alpha^{(v)} = \sum_{\ell=0}^k d_{k\ell}(v) (\rho \partial_\rho) x_\ell = \sum_{\ell=0}^k d_{k\ell}(v) \sum_{k_1 \dots k_\ell=1}^v \rho \partial_\rho \alpha_{k_1 \dots k_\ell}^{(v)} = \\ &= \sum_{\ell=0}^k d_{k\ell}(v) \sum_{k_1 \dots k_\ell=1}^v \left\{ \frac{1}{2} (v+2\ell) \alpha_{k_1 \dots k_\ell}^{(v)} - \frac{1}{2} \sum_{k_{\ell+1}=1}^v \alpha_{k_1 \dots k_{\ell+1}}^{(v)} \right\} \\ &= \sum_{\ell=0}^k \left(\frac{v}{2} + \ell \right) d_{k\ell}(v) \sum_{k_1 \dots k_\ell=1}^v \alpha_{k_1 \dots k_\ell}^{(v)} - \frac{1}{2} \sum_{\ell=0}^k d_{k\ell}(v) \sum_{k_1 \dots k_{\ell+1}=1}^v \alpha_{k_1 \dots k_{\ell+1}}^{(v)} \\ &= \sum_{\ell=0}^k \left[\left(\frac{v}{2} + \ell \right) d_{k\ell}(v) - \frac{1}{2} d_{k(\ell-1)} \right] x_\ell. \end{aligned} \quad (4.40)$$

The argument is complete provided we are able to show that $d_{k\ell}(v)$ fulfills the recurrence

$$d_{(k+1)\ell}(v) = \left(\frac{v}{2} + \ell \right) d_{k\ell}(v) - \frac{1}{2} d_{k(\ell-1)}(v). \quad (4.41)$$

To this aim, it is sufficient to make use of the basic recursive formulae $\left\{ \begin{matrix} n+1 \\ m \end{matrix} \right\} = m \left\{ \begin{matrix} n \\ m \end{matrix} \right\} + \left\{ \begin{matrix} n \\ m-1 \end{matrix} \right\}$ and $\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}$. We detail the algebra for the sake of completeness:

$$\begin{aligned} d_{(k+1)\ell}(v) &= \sum_{t=\ell}^{k+1} \frac{(-1)^\ell}{2^t} \phi_{t-\ell} \left\{ \begin{matrix} k+1 \\ t \end{matrix} \right\} \binom{t}{\ell} \\ &= \sum_{t=\ell}^{k+1} \frac{(-1)^\ell}{2^t} \phi_{t-\ell} t \left\{ \begin{matrix} k \\ t \end{matrix} \right\} \binom{t}{\ell} + \sum_{t=\ell}^{k+1} \frac{(-1)^\ell}{2^t} \phi_{t-\ell} \left\{ \begin{matrix} k \\ t-1 \end{matrix} \right\} \binom{t}{\ell} \\ &= \sum_{t=\ell}^{k+1} \frac{(-1)^\ell}{2^t} \phi_{t-\ell} t \left\{ \begin{matrix} k \\ t \end{matrix} \right\} \binom{t}{\ell} + \sum_{t=\ell-1}^k \frac{(-1)^\ell}{2^{t+1}} \phi_{t-\ell+1} \left\{ \begin{matrix} k \\ t \end{matrix} \right\} \binom{t+1}{\ell} \end{aligned}$$

$$\begin{aligned}
&= \sum_{t=\ell}^k \frac{(-1)^\ell}{2^t} \phi_{t-\ell} t \left\{ \begin{matrix} k \\ t \end{matrix} \right\} \binom{t}{\ell} + \sum_{t=\ell-1}^k \frac{(-1)^\ell}{2^{t+1}} \phi_{t-\ell+1} \left\{ \begin{matrix} k \\ t \end{matrix} \right\} \binom{t}{\ell} \\
&+ \sum_{t=\ell-1}^k \frac{(-1)^\ell}{2^{t+1}} \phi_{t-\ell+1} \left\{ \begin{matrix} k \\ t \end{matrix} \right\} \binom{t}{\ell-1} = \sum_{t=\ell}^k \frac{(-1)^\ell}{2^t} \phi_{t-\ell} t \left\{ \begin{matrix} k \\ t \end{matrix} \right\} \binom{t}{\ell} \\
&+ \sum_{t=\ell-1}^k \frac{(-1)^\ell}{2^{t+1}} [v - 2(t - \ell)] \phi_{t-\ell} \left\{ \begin{matrix} k \\ t \end{matrix} \right\} \binom{t}{\ell} - \frac{1}{2} d_{k(\ell-1)}(v) = \left(\frac{v}{2} + \ell \right) d_{k\ell}(v) - \frac{1}{2} d_{k(\ell-1)}(v). \tag{4.42}
\end{aligned}$$

□

In view of eq. (4.33), it is no surprise that the coefficients $d_{k\ell}(v)$ embody Stirling numbers of the second kind. Recall indeed that Stirling numbers of the first and second kind are related to each others by the inversion identity

$$\sum_{t=0}^{\max\{j,k\}} (-1)^{t-k} \left\{ \begin{matrix} t \\ j \end{matrix} \right\} \left[\begin{matrix} j \\ t \end{matrix} \right] = \delta_{jk}. \tag{4.43}$$

From Proposition 4.2 and eq. (4.43), it follows

Proposition 4.3. *For $k \geq 1$, we have*

$$\eta_k^{(v)}(\rho; \lambda) = \sum_{\ell=0}^k c_{k\ell}(v) \sum_{k_1 \dots k_\ell=1}^v \frac{\alpha_{k_1 \dots k_\ell}^{(v)}(\rho; \lambda)}{\alpha^{(v)}(\rho; \lambda)}, \tag{4.44}$$

with the coefficients $c_{k\ell}(v)$ defined as

$$c_{k\ell}(v) = \begin{cases} \frac{(-1)^\ell}{2^k} \frac{v!!}{[v - 2(k - \ell)]!!} \binom{k}{\ell}, & 0 \leq \ell \leq k, \\ 0, & \text{otherwise.} \end{cases} \tag{4.45}$$

Proof. We have all the necessary ingredients to carry out the proof. Again, we detail the algebra for the reader's convenience:

$$\begin{aligned}
\eta_k^{(v)} &= \frac{1}{\alpha^{(v)}} \rho^k \partial_\rho^k \alpha^{(v)} = \frac{1}{\alpha^{(v)}} \sum_{\ell=1}^k (-1)^{k-\ell} \left[\begin{matrix} k \\ \ell \end{matrix} \right] (\rho \partial_\rho)^\ell \alpha^{(v)} = \frac{1}{\alpha^{(v)}} \sum_{\ell=1}^k (-1)^{k-\ell} \left[\begin{matrix} k \\ \ell \end{matrix} \right] \sum_{m=0}^{\ell} d_{\ell m}(v) x_m \\
&= \frac{1}{\alpha^{(v)}} \sum_{\ell=1}^k (-1)^{k-\ell} \left[\begin{matrix} k \\ \ell \end{matrix} \right] \sum_{m=0}^{\infty} \sum_{t=m}^{\infty} \frac{(-1)^m}{2^t} \phi_{t-m} \left\{ \begin{matrix} \ell \\ t \end{matrix} \right\} \binom{t}{m} x_m \\
&= \frac{1}{\alpha^{(v)}} \sum_{m=0}^{\infty} \sum_{t=m}^{\infty} \frac{(-1)^m}{2^t} \phi_{t-m} \sum_{\ell=1}^k (-1)^{k-\ell} \left[\begin{matrix} k \\ \ell \end{matrix} \right] \left\{ \begin{matrix} \ell \\ t \end{matrix} \right\} \binom{t}{m} x_m \\
&= \frac{1}{\alpha^{(v)}} \sum_{m=0}^{\infty} \sum_{t=m}^{\infty} \frac{(-1)^m}{2^t} \phi_{t-m} \delta_{kt} \binom{t}{m} x_m = \frac{1}{\alpha^{(v)}} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^k} \phi_{k-m} \binom{k}{m} x_m \\
&= \frac{1}{\alpha^{(v)}} \sum_{m=0}^k \frac{(-1)^m}{2^k} \phi_{k-m} \binom{k}{m} x_m. \tag{4.46}
\end{aligned}$$

Note that on the second line above we could extend the upper bound of the sums over m and t from ℓ to ∞ thanks to the property $\left\{ \begin{smallmatrix} a \\ b \end{smallmatrix} \right\} = 0$ if $b > a$. This in turn allowed us to perform the sum exchange on the third line. By a similar argument, on the last line we could reduce the upper bound of the sum over m from ∞ to k . \square

Obviously, computing eq. (4.44) becomes increasingly demanding for larger values of k . Nonetheless, many contributions to the sum on the *r.h.s.* coincide. In particular, all Gaussian integrals with the same index multiplicities contribute equally, thus we can recast eq. (4.44) to the computationally cheaper expression

$$\eta_k^{(v)}(\rho; \lambda) = \sum_{\ell=0}^k c_{k\ell}(v) \sum_{m_1 \dots m_v=0}^{\ell} \binom{\ell}{m_1, \dots, m_v} \delta_{\ell, m_1 + \dots + m_v} \frac{\alpha_{1:m_1 \dots v:m_v}^{(v)}(\rho; \lambda)}{\alpha^{(v)}(\rho; \lambda)}. \quad (4.47)$$

4.5 Asymptotic sign of η_k

In this and next subsection we put the combinatorial approach on hold and work on the integral representation of the coefficient functions. A first property which turns out to be essential to the last part of the paper concerns the sign assumed by η_k as $\rho \rightarrow \infty$. In regard to this, we state the following

Proposition 4.4. *As $\rho \rightarrow \infty$ the sign of η_k becomes independent of v and λ . In particular, we have*

$$\lim_{\rho \rightarrow \infty} \text{sign } \eta_k^{(v)}(\rho; \lambda) = (-1)^{k-1}. \quad (4.48)$$

Proof. We first express α in spherical coordinates, *i.e.* we perform the change of integration variable $x = ru$ in eq. (1.4), with $r = \|x\|$, $u \in \partial\mathcal{B}_v(1)$ and $\partial\mathcal{B}_v(1) = \{z \in \mathbb{R}^v : \|z\| = 1\}$ (in the sequel we write $d^v x = r^{v-1} dr du$; here du embodies the angular part of the spherical Jacobian and the differentials of $(v-1)$ angles). Thus, we have

$$\alpha^{(v)}(\rho; \lambda) = \frac{1}{2\Gamma(v/2)|\Lambda|^{1/2}} \mathbb{M} \left[\int_0^{\sqrt{\rho}} dr \, r^{v-1} \exp \left\{ -\frac{r^2 \mathcal{P}(u)}{2} \right\} \right], \quad \mathcal{P}(u) = u^\top \Lambda^{-1} u, \quad (4.49)$$

with \mathbb{M} representing the uniform average operator on $\partial\mathcal{B}_v(1)$, namely

$$\mathbb{M}[g] = \frac{\Gamma(v/2)}{2\pi^{v/2}} \int_{\partial\mathcal{B}_v(1)} du \, g(u). \quad (4.50)$$

In order to compute η_k , we differentiate α under the integral sign. The first derivative evaluates the radial integral at its upper limit, while the remaining $k-1$ ones distribute according to the chain rule of differentiation. Explicitly, we have

$$\begin{aligned} \rho^k \partial_\rho^k \alpha^{(v)}(\rho; \lambda) &= \frac{\rho^k}{2^{v/2} \Gamma(v/2) |\Lambda|^{1/2}} \mathbb{M} \left[\partial_\rho^{k-1} \left(\rho^{v/2-1} \exp \left\{ -\rho \frac{\mathcal{P}(u)}{2} \right\} \right) \right] \\ &= \frac{(-1)^{k-1} \rho^{v/2}}{2^{v/2} \Gamma(v/2) |\Lambda|^{1/2}} \mathbb{M} \left[\sum_{\ell=0}^{k-1} \binom{k-1}{\ell} (-\phi)^{\bar{\ell}} \left(\frac{\rho \mathcal{P}(u)}{2} \right)^{k-1-\ell} \exp \left\{ -\frac{\rho \mathcal{P}(u)}{2} \right\} \right]_{\phi=v/2-1} \\ &= \frac{(-1)^{k-1} \rho^{v/2}}{2^{v/2} \Gamma(v/2) |\Lambda|^{1/2}} \mathbb{M} \left[\mathcal{Q}_{k-1} \left(\frac{\rho \mathcal{P}(u)}{2}, -\phi \right) \exp \left\{ -\frac{\rho \mathcal{P}(u)}{2} \right\} \right]_{\phi=v/2-1}. \end{aligned} \quad (4.51)$$

Here $x^{\bar{n}} \equiv x(x+1) \dots (x+n-1)$ denotes the n^{th} raising factorial of x and

$$\mathcal{Q}_k(x, a) = \sum_{\ell=0}^k \binom{k}{\ell} a^{\bar{\ell}} x^{k-\ell} \quad (4.52)$$

is a polynomial in x of k^{th} degree, differing from a Newton polynomial for the presence of $a^{\bar{\ell}}$ in place of a^{ℓ} . We note that the coefficient of the leading term of $Q_k(x, a)$ is $\binom{k}{0}a^{\bar{0}} = 1$. It follows that

$$\rho^k \partial_{\rho}^k \alpha^{(v)}(\rho; \lambda) \stackrel{\rho \rightarrow \infty}{\sim} \frac{(-1)^{k-1}}{\Gamma(\phi+1)|\Lambda|^{1/2}} \left(\frac{\rho}{2}\right)^{\phi+k} \mathbb{M} \left[\mathcal{P}(u)^{k-1} \exp \left\{ -\frac{\rho \mathcal{P}(u)}{2} \right\} \right] \quad (4.53)$$

Eq. (4.48) follows from the positiveness of $\mathcal{P}(u)$. \square

Since $\mathcal{P}(u) > \lambda_{\max}^{-1}$, being $\lambda_{\max} = \max_k \{\lambda_k\}$, we obtain as a by-product an estimate of the exponential damping of η_k , namely

$$\left| \rho^k \partial_{\rho}^k \alpha^{(v)}(\rho; \lambda) \right| < \frac{\rho^{v/2}}{2^{v/2} \Gamma(v/2) |\Lambda|^{1/2}} \left| \mathbb{M} \left[\mathcal{Q}_{k-1} \left(\frac{\rho \mathcal{P}(u)}{2}, -\phi \right) \right]_{\phi=v/2-1} \right| e^{-\rho/2 \lambda_{\max}}. \quad (4.54)$$

4.6 A convergence estimate for the expansion

We come back to the issue raised at the beginning of this section: is the weak truncation expansion convergent? It is not difficult to see that the answer lies specifically on the behavior of η_k as a function of k . Eq. (4.54) shows that the relevant information is brought by $\mathcal{Q}_{k-1}(\rho \mathcal{P}(u)/2, -\phi)$, particularly by its relative minima/maxima. As k increases, the position of the latter shifts towards larger and larger values of ρ , while their absolute size increases. In other words, however we choose a reference scale $\tilde{\rho} > 0$ we always find \tilde{k} such that $\arg \max\{|\eta_k(\rho; \lambda)|\} > \tilde{\rho}$ and $\max\{|\eta_k(\rho; \lambda)|\} > \max\{|\eta_{\tilde{k}}(\rho; \lambda)|\}$ for $k > \tilde{k}$. For this reason, the convergence issue reduces to quantify the increase rate of η_k as a function of k .

More quantitatively, we first notice the inequality $\gamma(a, x) < x^a/a$. In order to prove this, we observe that a convenient representation of the lower incomplete gamma function is provided by (see for instance sect. 6 of ref. [7])

$$\gamma(a, x) = \frac{1}{a} x^a e^{-x} M(1, 1+a, x), \quad (4.55)$$

where

$$M(a, b, x) = {}_1F_1(a; b; x) = \sum_{n=0}^{\infty} \frac{a^{\bar{n}} x^n}{b^{\bar{n}} n!} \quad (4.56)$$

is the confluent hypergeometric function originally introduced by Kummer. Since $1^{\bar{n}} = n!$ and $(1+a)^{\bar{n}} = (1+a)(2+a)\dots(n+a) > n!$ if $a > 0$, it follows

$$M(1, 1+a, x) = \sum_{n=0}^{\infty} \frac{1^{\bar{n}} x^n}{(1+a)^{\bar{n}} n!} = \sum_{n=0}^{\infty} \frac{x^n}{(1+a)^{\bar{n}}} < \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x. \quad (4.57)$$

We can use eq. (4.2) and the above inequality to establish an upper bound to the 1-dimensional Gaussian integrals, namely

$$\alpha_{n:k}^{(1)}(\rho; \lambda_k) < \frac{1}{\sqrt{2\pi k}} \left(\frac{\rho}{\lambda_n} \right)^{k+1/2}. \quad (4.58)$$

Now, let us denote by $\mathcal{X}_{n:k}$ the weak truncation expansion of $\alpha_{n:k}$, *i.e.*

$$\mathcal{X}_{n:k}^{(v)}(\rho; \lambda) = \alpha^{(v-1)}(\rho; \lambda_{(n)}) \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \left(\frac{\lambda_n}{\rho} \right)^p \alpha_{n:(k+p)}^{(1)}(\rho; \lambda_n) \eta_p^{(v-1)}(\rho; \lambda_{(n)}). \quad (4.59)$$

In view of eq. (4.58), an absolute estimate to $\mathcal{X}_{n:k}$ is given by

$$\begin{aligned} |\mathcal{X}_{n:k}^{(v)}(\rho; \lambda)| &< \sum_{p=0}^{\infty} \frac{1}{p!} \left(\frac{\lambda_n}{\rho} \right)^p \alpha_{n:(k+p)}^{(1)}(\rho; \lambda_n) \alpha^{(v-1)}(\rho; \lambda_{(n)}) \left| \eta_p^{(v-1)}(\rho; \lambda_{(n)}) \right| \\ &< \frac{1}{\sqrt{2\pi}} \left(\frac{\rho}{\lambda_n} \right)^{k+1/2} \sum_{p=0}^{\infty} \frac{1}{p!} \frac{1}{(p+k)} \left| \rho^p \partial_{\rho}^p \alpha^{(v-1)}(\rho; \lambda_{(k)}) \right|. \end{aligned} \quad (4.60)$$

From the above inequality we see that a less than factorial growth of η_p with p would make the *r.h.s.* of eq. (4.60) convergent. In order to estimate $|\rho^p \partial_{\rho}^p \alpha|$, we make use of eq. (4.51). Since $\partial \mathcal{B}_v(1)$ is a compact domain, we can get rid of the angular average by defining

$$u^* = \arg \max_{u \in \partial \mathcal{B}_v(1)} \left\{ \left| \mathcal{Q}_{p-1} \left(\frac{\rho \mathcal{P}(u)}{2}, -\phi \right) \right| \exp \left\{ -\frac{\rho \mathcal{P}(u)}{2} \right\} \right\}, \quad (4.61)$$

whence it follows

$$\left| \rho^p \partial_{\rho}^p \alpha^{(v)}(\rho; \lambda) \right| \leq \frac{\rho^{v/2}}{2^{v/2} \Gamma(v/2) |\Lambda|^{1/2}} \left| \mathcal{Q}_{p-1} \left(\frac{\rho \mathcal{P}(u^*)}{2}, -\phi \right) \right| \exp \left\{ -\frac{\rho \mathcal{P}(u^*)}{2} \right\}. \quad (4.62)$$

We have already noticed that $\mathcal{P}(u) > \lambda_{\max}^{-1}$ for all $u \in \partial \mathcal{B}_v(1)$. If we also consider that $|\Lambda|^{1/2} > \lambda_{\min}^{v/2}$, being $\lambda_{\min} = \min_k \{\lambda_k\}$, then we have $[\mathcal{P}(u)^{v/2} |\Lambda|^{1/2}]^{-1} < (\lambda_{\max}/\lambda_{\min})^{1/2}$. Multiplying and dividing the *r.h.s.* of eq. (4.62) by $\mathcal{P}(u^*)^{v/2}$ leads to

$$\begin{aligned} \left| \rho^p \partial_{\rho}^p \alpha^{(v)}(\rho; \lambda) \right| &< \frac{1}{\Gamma(v/2)} \left(\frac{\lambda_{\max}}{\lambda_{\min}} \right)^{v/2} \left(\frac{\rho \mathcal{P}(u^*)}{2} \right)^{v/2} \left| \mathcal{Q}_{p-1} \left(\frac{\rho \mathcal{P}(u^*)}{2}, -\phi \right) \right| \exp \left\{ -\frac{\rho \mathcal{P}(u^*)}{2} \right\} \\ &< \frac{1}{\Gamma(v/2)} \left(\frac{\lambda_{\max}}{\lambda_{\min}} \right)^{v/2} \max_{x \in \mathbb{R}_+} \left\{ x^{v/2} |\mathcal{Q}_{p-1}(x, -\phi)| e^{-x} \right\}. \end{aligned} \quad (4.63)$$

Accordingly, we obtain the estimate

$$|\mathcal{X}_{n:k}^{(v)}(\rho; \lambda)| < \frac{1}{\sqrt{2\pi} \Gamma((v-1)/2)} \left(\frac{\tilde{\lambda}_{\max}}{\tilde{\lambda}_{\min}} \right)^{(v-1)/2} \left(\frac{\rho}{\lambda_n} \right)^{k+1/2} \sum_{p=1}^{\infty} \frac{\mathcal{C}^{(v)}(p)}{p}, \quad (4.64)$$

where we have set $\tilde{\lambda}_{\min} = \min_{j \neq n} \{\lambda_j\}$, $\tilde{\lambda}_{\max} = \max_{j \neq n} \{\lambda_j\}$ and

$$\mathcal{C}^{(v)}(p) = \frac{1}{p} \max_{x \in \mathbb{R}_+} \left\{ \left| \sum_{\ell=0}^{p-1} \frac{(-\phi^*)^{\bar{\ell}} x^{p-\ell-\phi^*}}{\ell!(p-1-\ell)!} \right| e^{-x} \right\}_{\phi^*=(v-3)/2}. \quad (4.65)$$

It will be observed that in order to arrive at eq. (4.64), we have gone through a rather long inequality chain, so it is not clear whether the resulting upper bound is finite. For the sum on the *r.h.s.* to be convergent, it suffices that $\exists \epsilon > 0$, $A > 0$ and p_0 such that $\mathcal{C}(p) < A p^{-\epsilon} \equiv m(p; A, \epsilon)$ for $p > p_0$. If this holds true, then we have $\sum_{p=1}^{\infty} \mathcal{C}(p)/p < A \zeta(1 + \epsilon) + \text{const.}$, where ζ denotes the Riemann zeta function. In order to evaluate $\mathcal{C}(p)$ analytically, we need to solve a polynomial equation of degree $(p - \phi^*)$ for $p \gg 1$. An alternative approach is to compute $\mathcal{C}(p)$ numerically for a set of sufficiently large values of p and then try to fit data to a model such as $m(p; A, \epsilon)$, with the parameters A and ϵ depending in general on v . In Fig. 5 (right) we plot numerical determinations of $\mathcal{C}(p)$ for $v = 2, \dots, 6$. We observe that $\mathcal{C}(p)$ is monotonic decreasing for $v \leq 5$ and monotonic increasing for $v \geq 6$. In Fig. 5 (left) we report the values of the fitted parameters A and ϵ together with the corresponding χ^2 -values (the range chosen for all fits is $p \in [50, 100]$). Our numerical experiments suggest that the weak truncation expansion converges uniformly in ρ at least for $v \leq 5$. Nevertheless, the argument is not conclusive, since it assumes that we can extrapolate the fitting model to $p \rightarrow \infty$, which is not mathematically rigorous...

v	A	ϵ	χ^2/ndf
2	0.522	0.734	1.08×10^{-11}
3	0.396	0.499	9.86×10^{-13}
4	0.361	0.278	5.63×10^{-10}
5	0.375	0.068	2.16×10^{-8}
6	0.227	-0.309	3.30×10^{-7}

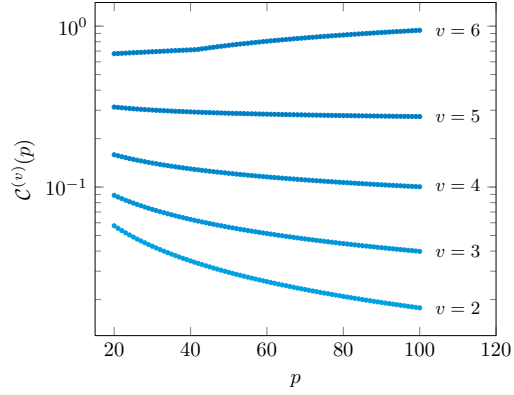


Fig. 5 – Numerical computation of $\mathcal{C}^{(v)}(p)$ for $v = 2, \dots, 6$. We fit each curve on the right to a model $m(p; A, \epsilon) = Ap^{-\epsilon}$, with A and ϵ depending in general on v . Fitted parameters are reported on the left, together with the corresponding χ^2 -values.

5 Variance reduction in the regime of weak truncation

If we look at eq. (4.21), we see that the next-to-leading contribution to the rescaled variance Γ_{nn} is the sum of a few ratios of 1-dimensional Gaussian integrals, all proportional to η_1 . The subsequent terms of the expansion have an increasingly complex structure. Each power of λ_n/ρ couples to several products of coefficient functions η_k , always combined so as to give the correct power counting. If $f(\rho; \lambda)$ is a generic observable, its weak truncation expansion reads

$$f(\rho; \lambda) = \sum_{q=0}^{\infty} (-1)^q \left(\frac{\lambda_n}{\rho} \right)^q \sum_{\underline{e} \in \mathcal{S}_q^q} \Xi_f^{(q; \underline{e})}(\rho, \lambda) \eta_0^{(v-1)}(\rho; \lambda_{(n)})^{e_0} \dots \eta_q^{(v-1)}(\rho; \lambda_{(n)})^{e_q}. \quad (5.1)$$

We denote by $\underline{e} = \{e_0, \dots, e_q\}$ the exponents of η_0, \dots, η_q and by \mathcal{S}_q^m the set of all possible \underline{e} 's corresponding to an overall power counting m , namely

$$\mathcal{S}_q^m = \left\{ \underline{e} \in \mathbb{N}_0^{q+1} : \mathcal{P}_q(\underline{e}) = m \right\}, \quad (5.2)$$

with the power counting function $\mathcal{P}_q(\underline{e})$ defined as

$$\mathcal{P}_q(\underline{e}) = \sum_{k=1}^q k e_k. \quad (5.3)$$

Note that if $m < q$ and $\underline{e} \in \mathcal{S}_q^m$, then $e_{m+1} = \dots = e_q = 0$. In this case, we interpret $\Xi_f^{(m; \underline{e})}$ as $\Xi_f^{(m; \{e_0, \dots, e_m\})}$. Recall also that since $\eta_0 = 1$, e_0 never contributes to the power counting. For later convenience, we define $\bar{e} = \{e_1, \dots, e_q\}$. If $\underline{e} \in \mathcal{S}_q^m$, with abuse of notation we also write $\bar{e} \in \mathcal{S}_q^m$. Strictly speaking, the presence of η_0 in eq. (5.1) is necessary in order to properly take into account the leading order of the expansion, namely

$$\lim_{\rho \rightarrow \infty} f(\rho; \lambda) = \lim_{\rho \rightarrow \infty} \sum_{e_0=0}^{\infty} \Xi_f^{(0; \{e_0\})}(\rho; \lambda). \quad (5.4)$$

The sum over e_0 in eq. (5.1) extends in principle from 0 to ∞ . We use e_0 to enumerate all contributions proportional to $\eta_1^{e_1} \dots \eta_q^{e_q}$. Accordingly, the information concerning the maximum value taken by e_0 is hidden

within $\Xi_f^{(q;\underline{e})}$. We finally stress that $\Xi_f^{(q;\underline{e})}$ depends in general upon all the components of λ , yet in the specific cases of Γ_{nn} and Δ_n (which are the ones we are interested in) it depends only upon λ_n .

Suppose now that $f(\rho; \lambda)$ and $g(\rho; \lambda)$ are two observables, which we expand according to eq. (5.1). It is not difficult to prove that the convolution rules needed to obtain the expansion of the algebraic combinations $f + g$ and $f \cdot g$ are similar to the Fourier's ones. Specifically, we have

$$\Xi_{f+g}^{(q;\underline{e})} = \Xi_f^{(q;\underline{e})} + \Xi_g^{(q;\underline{e})}, \quad (5.5)$$

and

$$\Xi_{f \cdot g}^{(q;\underline{e})} = \sum_{\ell, m=0}^q \delta_{q, \ell+m} \sum_{\underline{c} \in \mathcal{S}_q^\ell} \sum_{\underline{d} \in \mathcal{S}_q^m} \delta_{\underline{e}, \underline{c}+\underline{d}} \Xi_f^{(\ell; \underline{c})} \Xi_g^{(m; \underline{d})}, \quad (5.6)$$

where $\delta_{\underline{c}, \underline{d}} = \prod_{k=0}^q \delta_{c_k, d_k}$ is a vector generalization of the Kronecker symbol. Eqs. (5.5) and (5.6) are sufficient to derive the weak truncation expansion of Δ_n to all orders and to consequently prove our main result, represented by the following

Theorem 5.1. *As $\rho \rightarrow \infty$, the sign of the coefficient function $\Xi_{\Delta_n}^{(q;\underline{e})}(\rho; \lambda_n)$ is given by*

$$\lim_{\rho \rightarrow \infty} \sum_{\underline{e}_0} \Xi_{\Delta_n}^{(q;\underline{e})}(\rho; \lambda_n) = (-1)^{\sum_{k=1}^q e_k - 1}. \quad (5.7)$$

This, in conjunction with eq. (4.48), implies that all terms of the weak truncation expansion of Δ_n become negative at sufficiently large ρ .

Proof. We proceed in subsequent steps. First of all, we observe that Δ_n can be written as

$$\Delta_n(\rho; \lambda) = \frac{\lambda_n^2}{\rho^2} \mathfrak{D}_n \cdot \mathfrak{D}_d; \quad \begin{cases} \mathfrak{D}_n = \alpha_{nn}^{(v)} \alpha^{(v)} - \alpha_n^{(v)} \alpha_n^{(v)} - 2 \alpha_n^{(v)} \alpha^{(v)}, \\ \mathfrak{D}_d = [\alpha^{(v)}]^{-2}. \end{cases} \quad (5.8)$$

To work out \mathfrak{D}_n , we first review the expansion of $\alpha_{n:k}$. Explicitly, we have

$$\begin{aligned} \alpha_{n:k}^{(v)} &= \sum_{q=0}^{\infty} \frac{(-1)^q}{q!} \left(\frac{\lambda_n}{\rho} \right)^q \alpha_{n:(k+q)}^{(1)} \alpha^{(v-1)} \eta_q^{(v-1)} \\ &= \sum_{q=0}^{\infty} \frac{(-1)^q}{q!} \left(\frac{\lambda_n}{\rho} \right)^q \alpha_{n:(k+q)}^{(1)} \alpha^{(v-1)} \sum_{\underline{e} \in \mathcal{S}_q^q} \left(\prod_{i=0}^{q-1} \delta_{e_i, 0} \right) \delta_{e_q, 1} [\eta_0^{(v-1)}]^{e_0} \dots [\eta_q^{(v-1)}]^{e_q}. \end{aligned} \quad (5.9)$$

Hence, we deduce

$$\Xi_{\alpha_{n:k}}^{(q;\underline{e})} = \frac{1}{q!} \alpha_{n:(k+q)}^{(1)} \alpha^{(v-1)} \left(\prod_{i=0}^{q-1} \delta_{e_i, 0} \right) \delta_{e_q, 1}. \quad (5.10)$$

This allows us to derive the expansion of the product $\alpha_{n:r} \alpha_{n:s}$. From eq. (5.6) it follows

$$\begin{aligned} \Xi_{\alpha_{n:r} \alpha_{n:s}}^{(q;\underline{e})} &= \sum_{\ell, m=0}^q \frac{\delta_{q, \ell+m}}{\ell! m!} \alpha_{n:(r+\ell)}^{(1)} \alpha_{n:(s+m)}^{(1)} [\alpha^{(v-1)}]^2 \\ &\cdot \sum_{\underline{c} \in \mathcal{S}_q^\ell} \sum_{\underline{d} \in \mathcal{S}_q^m} \delta_{\underline{e}, \underline{c}+\underline{d}} \left(\prod_{i=0}^{\ell-1} \delta_{c_i, 0} \right) \left(\prod_{k=0}^{m-1} \delta_{d_k, 0} \right) \delta_{c_\ell, 1} \delta_{d_m, 1}. \end{aligned} \quad (5.11)$$

The inner sums can be performed exactly. Indeed, thanks to the Kronecker symbols $\delta_{c_\ell,1}$ and $\delta_{d_m,1}$, non-vanishing contributions group according to whether $\ell = m$ or $\ell \neq m$, namely

$$\Xi_{\alpha_{n:r} \cdot \alpha_{n:s}}^{(q;\underline{e})} = [\alpha^{(v-1)}]^2 \sum_{\ell,m=0}^q \frac{\delta_{q,\ell+m}}{\ell! m!} \alpha_{n:(r+\ell)}^{(1)} \alpha_{n:(s+m)}^{(1)} \cdot \left[\delta_{\ell,m} \delta_{e_\ell,2} \prod_{i \neq \ell}^{0 \dots q} \delta_{e_i,0} + (1 - \delta_{\ell,m}) \delta_{e_\ell,1} \delta_{e_m,1} \prod_{i \neq \ell,m}^{0 \dots q} \delta_{e_i,0} \right]. \quad (5.12)$$

We see that the Kronecker symbol $\delta_{q,\ell+m}$ makes both groups of terms vanish unless $\mathcal{P}_q(\underline{e}) = q$, as intuitively understood. Conversely, the only elements $\underline{e} \in \mathcal{S}_q^q$ which result in a non-vanishing coefficient function $\Xi_{\alpha_{n:r} \cdot \alpha_{n:s}}^{(q;\underline{e})}$ are either those where two different exponents e_ℓ, e_m equal one (with $\ell + m = q$) while the others vanish, or those where $e_{q/2} = 2$ and $e_i = 0$ for $i \neq q/2$ (of course the latter contribute only when q is even). From eq. (5.12), we immediately obtain

$$\Xi_{\mathfrak{D}_n}^{(q;\underline{e})} = [\alpha^{(v-1)}]^2 \sum_{\ell,m=0}^q \frac{\delta_{q,\ell+m}}{\ell! m!} \left(\alpha_{n:(\ell+2)}^{(1)} \alpha_{n:m}^{(1)} - \alpha_{n:(\ell+1)}^{(1)} \alpha_{n:(m+1)}^{(1)} - 2 \alpha_{n:(\ell+1)}^{(1)} \alpha_{n:m}^{(1)} \right) \cdot \left[\delta_{\ell,m} \delta_{e_\ell,2} \prod_{i \neq \ell}^{0 \dots q} \delta_{e_i,0} + (1 - \delta_{\ell,m}) \delta_{e_\ell,1} \delta_{e_m,1} \prod_{i \neq \ell,m}^{0 \dots q} \delta_{e_i,0} \right]. \quad (5.13)$$

The above expression depends upon ρ essentially via the integrals in parentheses (the overall factor $[\alpha^{(v-1)}]^2$ is irrelevant to our aims). Since $\alpha_{n:r} \rightarrow (2r-1)!!$ as $\rho \rightarrow \infty$, we have $(\alpha_{n:(\ell+2)} \alpha_{n:m} - \alpha_{n:(\ell+1)} \alpha_{n:(m+1)} - 2 \alpha_{n:(\ell+1)} \alpha_{n:m}) \rightarrow (\ell-m)(2\ell+1)(2\ell-1)!!(2m-1)!!$. In particular, this quantity vanishes for $\ell = m$, thus making the first term in square brackets never contribute as $\rho \rightarrow \infty$. A little additional algebra yields

$$\lim_{\rho \rightarrow \infty} \Xi_{\mathfrak{D}_n}^{(q;\underline{e})}(\rho; \lambda) = 4 \sum_{\ell=0}^q \sum_{m=0}^{\ell-1} \delta_{q,\ell+m} (\ell-m)^2 \frac{(2\ell-1)!!}{\ell!} \frac{(2m-1)!!}{m!} \delta_{e_\ell,1} \delta_{e_m,1} \prod_{i \neq \ell,m}^{0 \dots q} \delta_{e_i,0}. \quad (5.14)$$

We notice that the *r.h.s.* of eq. (5.14) vanishes always for $e_0 \geq 2$, but not necessarily for $e_0 = 0$ or $e_0 = 1$.

As a second step, we work out \mathfrak{D}_d . To this aim, we first need to evaluate $\Xi_{\alpha^{-1}}^{(q;\underline{e})}$. As already done in sect. 4, we make use of the Taylor series $(1+x)^{-1} = \sum_{p=0}^{\infty} (-1)^p x^p$. From

$$\alpha^{(v)} = \alpha^{(1)} \alpha^{(v-1)} \cdot \left[1 + \sum_{q=1}^{\infty} \frac{(-1)^q}{q!} \left(\frac{\lambda_n}{\rho} \right)^q \frac{\alpha_{n:q}^{(1)}}{\alpha^{(1)}} \eta_q^{(v-1)} \right], \quad (5.15)$$

it follows

$$\begin{aligned} [\alpha^{(v)}]^{-1} &= [\alpha^{(1)} \alpha^{(v-1)}]^{-1} \sum_{p=0}^{\infty} (-1)^p \sum_{\ell_1 \dots \ell_p=1}^{\infty} \frac{(-1)^{\ell_1+\dots+\ell_p}}{\ell_1! \dots \ell_p!} \left(\frac{\lambda_n}{\rho} \right)^{\ell_1+\dots+\ell_p} \frac{\alpha_{n:\ell_1}^{(1)} \dots \alpha_{n:\ell_p}^{(1)}}{[\alpha^1]^p} \eta_{\ell_1}^{(v-1)} \dots \eta_{\ell_p}^{(v-1)} \\ &= [\alpha^{(1)} \alpha^{(v-1)}]^{-1} \sum_{q=0}^{\infty} (-1)^q \left(\frac{\lambda_n}{\rho} \right)^q \sum_{p=0}^q (-1)^p \sum_{\ell_1 \dots \ell_p=1}^{\infty} \delta_{q,\ell_1+\dots+\ell_p} \prod_{j=1}^p \left[\frac{1}{\ell_j!} \frac{\alpha_{n:\ell_j}^{(1)}}{\alpha^{(1)}} \eta_{\ell_j}^{(v-1)} \right]. \end{aligned} \quad (5.16)$$

On the second line we have reduced the upper limit of the sum over p from ∞ to q . The reason is that $\delta_{q,\ell_1+\dots+\ell_p} = 0$ for $q < p$, owing to $\ell_1 + \dots + \ell_p \geq p$. On expanding the sums over ℓ_1, \dots, ℓ_p , we see that all terms proportional to $\eta_1^{e_1} \dots \eta_q^{e_q}$ for some \bar{e} coincide. Since permutations of ℓ_1, \dots, ℓ_p corresponding to the

same \bar{e} give all the same contribution, the latter turns out to be multiplied by an overall numerical factor which is at most $p!$. Of course, permutations of equal indices contribute only once. Therefore, a correct counting of that factor amounts to the multinomial coefficient $p!/(e_1! \dots e_q!)$, with the constraint $\sum e_j = p$. In other words, we have

$$[\alpha^{(v)}]^{-1} = [\alpha^{(1)} \alpha^{(v-1)}]^{-1} \sum_{q=0}^{\infty} (-1)^q \left(\frac{\lambda_n}{\rho} \right)^q \sum_{\underline{e} \in \mathcal{S}_q^q} (-1)^{\sum e_k} \delta_{e_0,0} \left(\sum_{e_0, \dots, e_k} e_k \right) \prod_{j=0}^q \left[\frac{1}{j!} \frac{\alpha_{n;j}^{(1)}}{\alpha^{(1)}} \eta_j \right]^{e_j}, \quad (5.17)$$

and consequently

$$\Xi_{\alpha^{-1}}^{(q;\underline{e})} = (-1)^{\sum_{k=1}^q e_k} \delta_{e_0,0} \left(\sum_{e_0, \dots, e_k} e_k \right) \prod_{j=0}^q \left[\frac{1}{j!} \frac{\alpha_{n;j}^{(1)}}{\alpha^{(1)}} \right]^{e_j} [\alpha^{(1)} \alpha^{(v-1)}]^{-1}. \quad (5.18)$$

Now, we obtain $\Xi_{\mathfrak{D}_d}^{(q;\underline{e})}$ by convolving eq. (5.18) with itself. This yields

$$\Xi_{\mathfrak{D}_d}^{(q;\underline{e})} = \left\{ (-1)^{\sum_{k=1}^q e_k} \delta_{e_0,0} \prod_{j=1}^q \left[\frac{1}{j!} \frac{\alpha_{n;j}^{(1)}}{\alpha^{(1)}} \right]^{e_j} \Psi^{(q;\bar{e})} \right\} [\alpha^{(1)} \alpha^{(v-1)}]^{-2}, \quad (5.19)$$

with the coefficient $\Psi^{(p;\bar{e})}$ defined by

$$\Psi^{(p;\bar{e})} = \sum_{\ell, m=0}^p \delta_{p, \ell+m} \sum_{\bar{c} \in \mathcal{S}_q^\ell} \sum_{\bar{d} \in \mathcal{S}_q^m} \delta_{\bar{e}, \bar{c}+\bar{d}} \left(\sum_{c_1, \dots, c_q}^q c_k \right) \left(\sum_{d_1, \dots, d_q}^q d_k \right), \quad |\bar{e}| = q. \quad (5.20)$$

From eq. (5.19), it follows

$$\lim_{\rho \rightarrow \infty} \Xi_{\mathfrak{D}_d}^{(q;\underline{e})} = (-1)^{\sum_{k=1}^q e_k} \delta_{e_0,0} \prod_{j=1}^q \left[\frac{(2j-1)!!}{j!} \right]^{e_j} \Psi^{(q;\bar{e})}. \quad (5.21)$$

As a third step, we convolve eqs. (5.14) and (5.21). In this way we obtain $\Xi_{\Delta_n}^{(q;\underline{e})}$ directly in the limit $\rho \rightarrow \infty$. The algebra is just a little bit intricate, so we detail it. First of all,

$$\begin{aligned} \lim_{\rho \rightarrow \infty} \Xi_{\Delta_n}^{(q;\underline{e})} &= 4 \sum_{\ell, m=0}^q \delta_{q, \ell+m} \sum_{\underline{c} \in \mathcal{S}_q^\ell} \sum_{\underline{d} \in \mathcal{S}_q^m} \delta_{e_0, c_0} \delta_{\bar{e}, \bar{c}+\bar{d}} \left[\sum_{r=0}^{\ell} \sum_{s=0}^{r-1} \delta_{n, r+s} (r-s)^2 \frac{(2r-1)!!}{r!} \frac{(2s-1)!!}{s!} \right. \\ &\quad \cdot \delta_{c_r, 1} \delta_{c_s, 1} \prod_{i \neq r, s}^{0 \dots \ell} \delta_{c_i, 0} \left. \right] \cdot \left\{ (-1)^{\sum_{k=1}^m d_k} \prod_{k=1}^m \left[\frac{(2k-1)!!}{k!} \right]^{d_k} \Psi^{(m;\bar{d})} \right\}. \end{aligned} \quad (5.22)$$

Owing to the Kronecker symbols, we pay no price if we introduce a factor of $(-1)^{\sum_{k=0}^q c_k} \equiv 1$ within square brackets. For the same reason, we can also insert additional factors of $[(2k-1)!!/k!]^{c_k} \equiv 1$ for $k \neq r, s$ without pay. Hence,

$$\begin{aligned} \lim_{\rho \rightarrow \infty} \Xi_{\Delta_n}^{(q;\underline{e})} &= 4 (-1)^{\sum_{k=1}^q e_k} \prod_{k=1}^q \left[\frac{(2k-1)!!}{k!} \right]^{e_k} \sum_{\ell, m=0}^q \delta_{q, \ell+m} \sum_{\underline{c} \in \mathcal{S}_q^\ell} \sum_{\bar{d} \in \mathcal{S}_q^m} (-1)^{c_0} \delta_{e_0, c_0} \delta_{\bar{e}, \bar{c}+\bar{d}} \\ &\quad \cdot \sum_{r=0}^{\ell} \sum_{s=0}^{r-1} \delta_{\ell, r+s} (r-s)^2 \delta_{c_r, 1} \delta_{c_s, 1} \prod_{i \neq r, s}^{0 \dots \ell} \delta_{c_i, 0} \Psi^{(m;\bar{d})} \\ &= 4 (-1)^{\sum_{k=1}^q e_k} \delta_{\mathcal{P}_q(\underline{e}), q} \prod_{k=1}^q \left[\frac{(2k-1)!!}{k!} \right]^{e_k} \sum_{\ell, m=0}^q \delta_{q, \ell+m} \sum_{\underline{c} \in \mathcal{S}_q^\ell} (-1)^{c_0} \delta_{e_0, c_0} \\ &\quad \cdot \sum_{r=0}^{\ell} \sum_{s=0}^{r-1} \delta_{\ell, r+s} (r-s)^2 \delta_{c_r, 1} \delta_{c_s, 1} \prod_{i \neq r, s}^{0 \dots \ell} \delta_{c_i, 0} \Psi^{(m;\bar{e}-\bar{c})} \theta_{\underline{e}, \underline{c}}, \end{aligned} \quad (5.23)$$

with $\theta_{\bar{e}, \bar{c}} = \prod_{i=1}^q \theta_{e_i, c_i}$ being a vector generalization of the Heaviside function

$$\theta_{a,b} = \begin{cases} 1 & \text{if } a \geq b, \\ 0 & \text{otherwise.} \end{cases} \quad (5.24)$$

From eq. (5.23) it follows

$$\lim_{\rho \rightarrow \infty} \sum_{e_0=0}^{\infty} \Xi_{\Delta_n}^{(q; \underline{e})} = 4(-1)^{\sum_{k=1}^q e_k} \delta_{\mathcal{P}_q(\underline{e}), q} \prod_{k=1}^q \left[\frac{(2k-1)!!}{k!} \right]^{e_k} \left\{ \Omega_0^{(q; \bar{e})} - \Omega_1^{(q; \bar{e})} \right\}, \quad (5.25)$$

with

$$\Omega_0^{(q; \bar{e})} = \sum_{\ell, m=0}^q \delta_{q, \ell+m} \sum_{r=1}^{\ell-1} \sum_{s=1}^{r-1} \delta_{\ell, r+s} (r-s)^2 \theta_{e_s, 1} \theta_{e_r, 1} \Psi^{(m; \{e_1, \dots, e_s-1, \dots, e_r-1, \dots, e_\ell, \dots, e_q\})}, \quad (5.26)$$

and

$$\Omega_1^{(q; \bar{e})} = \sum_{\ell, m=0}^q \delta_{q, \ell+m} \ell^2 \theta_{e_\ell, 1} \Psi^{(m; \{e_1, \dots, e_\ell-1, \dots, e_q\})}. \quad (5.27)$$

In order to complete the proof, we need to show that $\Omega_0^{(q; \bar{e})} < \Omega_1^{(q; \bar{e})} \forall \bar{e} \in \mathcal{S}_q^q$. To this aim, we find it convenient to use a slightly different representation of $\Psi^{(p; \bar{e})}$, *viz.*

$$\Psi^{(p; \bar{e})} = \delta_{\mathcal{P}_q(\bar{e}), p} \sum_{n=1}^p \sum_{\bar{c} \in \mathcal{S}_q^n} \left(\sum_{k=1}^q c_k \right) \left(\sum_{k=1}^q (e_k - c_k) \right) \theta_{\bar{e}, \bar{c}}. \quad (5.28)$$

Suppose that $c_\ell \geq 1$ for some $1 \leq \ell \leq q$. Then,

$$\begin{aligned} & \theta_{e_\ell, 1} \Psi^{(q-\ell; \{e_1, \dots, e_\ell-1, \dots, e_q\})} \\ &= \delta_{\mathcal{P}_q(\bar{e}), q} \sum_{t=1}^q \sum_{\bar{c} \in \mathcal{S}_q^t} \left(\sum_{k=1}^q c_k \right) \left(\sum_{k=1}^q (e_k - c_k) - 1 \right) \theta_{e_n, c_n+1} \prod_{s \neq n}^{1 \dots q} \theta_{e_s, c_s}. \end{aligned} \quad (5.29)$$

On performing the change of variable $c_\ell \rightarrow c_\ell + 1$, we easily arrive at

$$\theta_{e_\ell, 1} \Psi^{(q-\ell; \{e_1, \dots, e_\ell-1, \dots, e_q\})} = \delta_{\mathcal{P}_q(\bar{e}), q} \sum_{t=\ell}^q \sum_{\bar{c} \in \mathcal{S}_q^t} \frac{c_\ell}{\sum_{k=1}^q c_k} \left(\sum_{k=1}^q c_k \right) \left(\sum_{k=1}^q (e_k - c_k) \right) \theta_{\bar{e}, \bar{c}}. \quad (5.30)$$

Analogously, for $e_s \geq 1$, $e_r \geq 1$ and $1 \leq s < r < q$, we have

$$\begin{aligned} & \theta_{e_s, 1} \theta_{e_r, 1} \Psi^{(q-s-r; \{e_1, \dots, e_s-1, \dots, e_r-1, \dots, e_q\})} \\ &= \delta_{\mathcal{P}_q(\bar{e}), q} \sum_{t=r+s}^q \sum_{\bar{c} \in \mathcal{S}_q^t} \frac{c_s c_r}{(\sum_{k=1}^q c_k) (\sum_{k=1}^q c_k - 1)} \left(\sum_{k=1}^q c_k \right) \left(\sum_{k=1}^q (e_k - c_k) \right) \theta_{\bar{e}, \bar{c}} \theta_{\sum_{k=1}^q c_k, 2}. \end{aligned} \quad (5.31)$$

Note that the lower limit of the sum over t in eq. (5.30) can be reduced from ℓ to 1, since $c_\ell = 0$ if $\bar{e} \in \mathcal{S}_q^t$ and $t < \ell$. The same cannot be done in eq. (5.31) without increasing the resulting sum. Therefore,

$$\Omega_1^{(q; \bar{e})} = \delta_{\mathcal{P}_q(\bar{e}), q} \sum_{t=1}^q \sum_{\bar{c} \in \mathcal{S}_q^t} \frac{\sum_{\ell=1}^q \ell^2 c_\ell}{\sum_{k=1}^q c_k} \left(\sum_{k=1}^q c_k \right) \left(\sum_{k=1}^q (e_k - c_k) \right) \theta_{\bar{e}, \bar{c}}, \quad (5.32)$$

and

$$\Omega_0^{(q;\bar{c})} \leq \delta_{\mathcal{P}_q(\bar{c}),q} \sum_{t=1}^q \sum_{\bar{c} \in \mathcal{S}_q^t} \frac{\sum_{\ell=1}^q \sum_{r=1}^{\ell-1} \sum_{s=1}^{r-1} \delta_{\ell,r+s} (r-s)^2 c_s c_r}{\left(\sum_{k=1}^q c_k\right) \left(\sum_{k=1}^q c_k - 1\right)} \cdot \binom{c_k}{c_1, \dots, c_q} \binom{\sum_{k=1}^q (e_k - c_k)}{e_1 - c_1, \dots, e_q - c_q} \theta_{\bar{c}, \bar{c}}. \quad (5.33)$$

Now, it is immediate to prove that

$$\sum_{\ell=1}^q \sum_{r=1}^{\ell-1} \sum_{s=1}^{r-1} \delta_{\ell,r+s} (r-s)^2 c_s c_r < \sum_{\ell=1}^q \ell^2 c_\ell \left(\sum_{k=1}^q c_k - 1 \right), \quad \forall \bar{c} \in \mathcal{S}_q^t. \quad (5.34)$$

Indeed, given $\bar{c} \in \mathcal{S}_q^t$ each non-vanishing contribution $c_s c_r > 0$ with $s < r$ is weighted by $(r-s)^2$ on the *l.h.s.* and by $(r^2 + s^2)$ on the *r.h.s.* The remaining terms on the *r.h.s.* are $\sum_{\ell=1}^{q-1} \ell^2 c_\ell (c_\ell - 1) \geq 0$ and $q^2 c_q (\sum_{k=1}^q c_k - 1) \geq 0$. This concludes the proof. \square

6 Concluding remarks

Conditioning a vector $X \sim \mathcal{N}_v(0, \Lambda)$ with $\Lambda = \text{diag}(\lambda)$ to a centered Euclidean ball $\mathcal{B}_v(\rho)$ of square radius ρ affects non-trivially the covariance matrix of its square components. Since the conditional moments of X cannot be calculated in closed-form, the only viable approach (besides numerical computation) to characterizing the truncational effects consists in establishing analytic bounds to the conditional correlations (variances and covariances) of the square components of X . Such estimates are also referred to in the literature as *square correlation inequalities*.

In this paper, we specifically focused on the conditional variances. In particular, our aim was proving eq. (1.2). The analyses presented in the previous sections go in this direction, yet they do not solve the problem in a conclusive way. The arguments proposed apply in the opposite regimes of strong and weak truncation. For $0 < \rho < 2\lambda_n$, eq. (1.2) is easily proved. A bigger effort is required for $\rho \gg \lambda_n$. Nothing is said regarding the intermediate region. We conclude with two major criticisms, representing at the same time an outlook of future research:

- the weak truncation region is not sharply defined: the asymptotic property stated by Theorem 5.1 is certainly sufficient to prove that the p^{th} order of the weak truncation expansion of Δ_n is negative at $\rho > \rho_p^*$ for some ρ_p^* , but the theorem does not provide any estimate of ρ_p^* . A better characterization of the coefficient functions η_k and Ξ_{Δ_n} far from the asymptotic regime would help identify precise conditions to extend the proof of eq. (1.2) to large yet finite values of ρ along the same lines of Theorem 5.1;
- we also lack a general proof of convergence of the weak truncation expansion. The argument presented in sect. 4 suggests uniform convergence in $v \leq 5$ dimensions, but it is based on a numerical estimate of the vanishing rate of the p^{th} term of the expansion, which cannot be legitimately extrapolated to $p \rightarrow \infty$.

The weak truncation expansion of a given observable f (built from Gaussian integrals $\alpha_{k\ell m \dots}$) is to all extents a perturbative expansion around the factorized value f takes as $\rho \rightarrow \infty$. As such, it is affected by the usual problems encountered with perturbative expansions. Having proved a property of Δ_n to all orders represents the main (non-trivial) contribution of the present paper.

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